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NEIGHBOURHOOD MANIFOLDS AND THEIR PARAMETRIZATION

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CONTENTS

The main purpose of this paper is the representation by algebraic varieties in projective space, of certain aggregates, in which each element is a sequence $\mathsf{P}_0 \dots \mathsf{P}_n$ of $n+1$ points consecutive on an algebraic (or algebroid) branch. Two types of aggregate are to be considered: $W_{r,n}$, of all sequences of $n+1$ points with a given origin P_0 in S_r; and $W^*_{r,n}$, of all such sequences with origin anywhere in S_r. The problem has been studied by various writers from 1901 onwards, and a complete solution for $n = 1, 2$ only was given in 1955. In this period no progress at all had been made for higher values of n, and its intractability led to conjectures that the aggregates in question were not irreducible, or had some other defects inhibiting the type of representation sought.

In the present paper a complete method is given of parametrizating both $W_{r,n}$ and $W^*_{r,n}$ for all values of r, n, the co-ordinates of a point of the model being expressed systematically in terms of certain invariants of an arbitrary branch through the sequence represented. These models are irreducible and non-singular, and their points are in one-one correspondence without exception with the sequences in the appropriate aggregates. In addition, the geometrical properties of the models are studied in some detail for $r = 2$ (the clarity of the picture obtained naturally fading off as n increases) and ($W_{2,3}$ having been dealt with in the author's short contribution to a symposium in honour of Bompiani) a complete geometrical description is given of $W_{2,4}$, $W_{3,3}$, and $W^*_{2,3}$, including the base and intersection theory on these.

The main importance claimed for the work is twofold: (i) it offers in theory (i.e. by a well defined method, which however involves sharply increasing labour with increase of n and of r) a complete solution of a problem which has proved intractable for 60 years; (ii) it gives detailed descriptions of some, and less full description of some others, of a family of varieties which have the same type of importance for local algebraic geometry that for instance Grassmannians have for line geometry, etc. It is possible also that the very ample system of invariants defined in the course of the parametrization of the models may prove to be of use in differential geometry (the formal power series from which they are obtained being interpreted as Taylor series) but this is a pure surmise, being rather outside the author's sphere of interest.

1. Definitions

Though the ideas of neighbouring points and proximate points are tolerably familiar, there does not seem to be yet a sufficiently consistent and generally accepted terminology in use to make it possible to begin any discussion of them without a few definitions.

We are concerned with r-dimensional projective space S_r over a ground field K, of which the only properties that we shall assume are that it is commutative and without characteristic. (The latter does not seem to be important most of the time, but we shall for simplicity assume it throughout.) A point in the ordinary sense, defined by a set of co-ordinates, whether in K or in some extension of K, will be called explicit; in contrast to this, a point whose only existence is in some neighbourhood of an explicit point will be called implicit. By the dilation of an explicit point P_0 is meant the birational transformation of S_r into an r-fold $V^{(1)}$ so that P_0 is replaced by an S_{r-1}, say $S^{(1)}$, projective image of the star of lines through P_0, the image of any curve through P_0 meeting $S^{(1)}$ in the point corresponding to the tangent at P_0. The implicit points of S_r whose images are explicit points of $S^{(1)}$ are in the first neighbourhood of P_0; and points in the $(n-1)$th neighbourhood of an explicit point having been defined, any implicit point $\mathsf{P}_n^{(1)}$ of $V^{(1)}$ which is in the $(n-1)$th neighbourhood of an explicit point of $S^{(1)}$ is the image of a point P_n in the nth neighbourhood of P_0. Every such point P_n uniquely determines a sequence $\mathsf{P}_0\mathsf{P}_1\ldots\mathsf{P}_n$, of which P_0 alone is explicit, and P_i in the first neighbourhood of P_{i-1} $(i = 1, \ldots, n)$; $\mathsf{P}_0, \ldots, \mathsf{P}_{n-1}$ we shall call the ancestors of P_n, and in particular P_{n-1} its parent and P_0 its explicit ancestor. For all $j < n$, P_n is in the $(n-j)$th neighbourhood of P_j; $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \ldots$ can be dilated in turn, each dilation making the next point of the sequence explicit. Any such sequence of points, in which each except the first is in the first neighbourhood of its immediate predecessor, will be called a consecutive sequence.

If P_i is any ancestor of P_j, P_j is said to be proximate to P_i if and only if when $\mathsf{P}_0, \ldots, \mathsf{P}_i$ are dilated in turn so that the first neighbourhood of P_i is exhibited as an explicit S_{r-1} the image of P_j is a point (whether explicit or implicit) of this S_{r-1}; if it is an explicit point P_j is directly, otherwise indirectly proximate to P_i. The points proximate to any given one are thus all those of its first neighbourhood directly, and certain points of its further neighbourhoods indirectly; every implicit point is directly proximate to its parent, and may or may not be indirectly proximate to some of its other ancestors. An implicit point that is not indirectly proximate to any other point is called free; in two dimensions a point that is indirectly proximate to one of its ancestors is usually called a satellite point, but in more dimensions this expression is perhaps less appropriate; we shall use the word unfree for any implicit point that is not free, i.e. that is proximate not only directly to its parent but indirectly to some one or more of its remoter ancestors.

Given any set of n points, explicit or implicit, in S_r, the aggregate of points proximate to all of them is either (i) empty, or (ii) abstractly equivalent to S_{r-n}, including all the implicit points of the latter; and the necessary and sufficient conditions for the second alternative are, that the n points all belong to a consecutive sequence (so that not more than one of them can be explicit) and that each of them is proximate (whether directly or indirectly) to all of the others that precede it in the sequence. It follows that in S_r no point can be proximate to more than r others, and that all points to which it is indirectly proximate are points to which its parent is also proximate, directly or indirectly. It also follows that in any consecutive sequence, the points proximate to a particular one are some one or more immediately following it in the sequence.

Every algebroid curve branch with origin in an explicit point P_0 defines uniquely a consecutive sequence $P_0 P_1 P_2 \ldots$, of any length, of points which are on the branch; the multiplicity of the branch in any point P_i of the sequence is the sum of its multiplicities in points proximate to P_i. It follows that the number of unfree points in the sequence is finite, i.e. all points of the sequence from a certain one onwards are free and simple. In particular, if the branch is simple, i.e. has no multiple points, every point of the sequence is free. We shall call a sequence free or unfree, according as all its points are free, or some unfree.

We can specify the indirect proximity relations between the points of a consecutive sequence by means of a type symbol consisting of the ordinal suffixes of all the points that are indirectly proximate to each one in turn (direct proximity does not need to be specified, as each point is directly proximate to its predecessor, and to no other). More than one that are indirectly proximate to the same point will be contained in brackets, the point to which they are all indirectly proximate having the suffix less by 2 than the first suffix in the bracket. Thus (234) 3 (45) will denote a sequence in which $P_2 P_3 P_4$ are all indirectly proximate to P_0, P_3 is also indirectly proximate to P_1, and $P_4 P_5$ are both indirectly proximate to P_2.

2. The neighbourhood manifolds

We shall denote by $W_{r,n}$ the total nth neighbourhood of an explicit point P_0 of S_r, i.e. the totality of implicit points P_n of S_r that are in the nth neighbourhood of P_0. This is really the same thing as the totality of consecutive sequences $P_0 \ldots P_n$ beginning with the same explicit point P_0. $W_{r,0}$ is the point P_0 itself, and $W_{r,1}$ is a birational image of S_{r-1}. (Throughout this investigation, when we speak of a birational image, transformation, etc., we shall mean, unless the contrary is stated, one without fundamental elements, so that the point to point correspondence is one-one without exception.) If, as it seems natural to expect, it proves possible to identify $W_{r,n}$ with some algebraic variety, this will have to be generated by a congruence of birational images of $W_{r,n-j}$, just one member of the congruence passing through each point of $W_{r,n}$, and the congruence itself being a birational image of $W_{r,j}$; since $W_{r,n}$ is the union of the $(n-j)$th neighbourhoods of all the points P_j of $W_{r,j}$, and each of these is indistinguishable from (and in fact when the ancestors of P_j are dilated becomes) the $(n-j)$th neighbourhood of an explicit point. This holds for all $j < n$. We shall express this situation briefly by saying that $W_{r,n}$ is a fibre space of $W_{r,n-j}$'s over $W_{r,j}$, and in particular of S_{r-1}'s over $W_{r,n-1}$, and also of $W_{r,n-1}$'s over S_{r-1}. Still assuming that $W_{r,n}$ is to be thought of as an algebraic variety at all, it is obvious that it must be $n(r-1)$-dimensional.

Similarly, we define $W^*_{r,n}$ to be the union of the nth neighbourhoods of all explicit points of S_r. $W^*_{r,0}$ is S_r itself, and if $W^*_{r,n}$ is to be identified with some algebraic variety it must be (in the sense in which are are using the expression) a fibre space of $W_{r,n}$'s over S_r, and also of $W_{r,n-j}$'s over $W^*_{r,j}$, and in particular of S_{r-1}'s over $W^*_{r,n-1}$. $W^*_{r,n}$ must thus be $r+n(r-1)$-dimensional.

The main purpose of this paper is to show how, by a consistent and unambiguous process, we can construct a model of $W_{r,n}$, and also one of $W^*_{r,n}$, each as an irreducible non-singular algebraic variety in a suitable projective space. These models have the following obviously desirable invariance properties, which indicate that the geometrical properties of the model truly represent essential properties of the aggregate represented: (i) every self transformation of S_r, regular at P_0, induces a self collineation on the model of $W_{r,n}$; (ii) every self collineation in S_r induces a self collineation on the model of $W^*_{r,n}$; (iii) every linear dual self transformation in S_2 induces a self collineation on the model of $W^*_{2,n}$—it will be seen later that this third property is not reasonably to be looked for except for $r=2$.

Our results include and extend those of other investigators, who have solved the problem in a succession of stages for $n \leqslant 2$ only. The whole question seems to have been first raised by Study (1901), as that of assigning co-ordinates to plane curve elements of the first and second orders, i.e. to what we here call sequences $\mathsf{P}_0\,\mathsf{P}_1$ and $\mathsf{P}_0\,\mathsf{P}_1\,\mathsf{P}_2$ in the plane. He pointed out that the former can be represented by the homogeneous co-ordinates (X) of the point P_0 and (U) of the line $\mathsf{P}_0\,\mathsf{P}_1$, separately homogeneous, and satisfying the bilinear invariant relation $(UX)=0$; and (by considering the net of conics through $\mathsf{P}_0\,\mathsf{P}_1\,\mathsf{P}_2$) that the latter can be represented by the same six co-ordinates (X), (U), still of course satisfying $(UX)=0$, but not now independently homogeneous; instead, all six can be multiplied by the same homogeneity factor, and either triple can be multiplied by a cube root of unity, leaving the other triple unchanged. It will be seen that (X), (U) can be identified with the tensors $\mathbf{u}^{\alpha}(\mathbf{D}^*)^{\frac{1}{3}}$, $\mathbf{q}_{\beta}$ of our § 21; and Study obtained a parametrization of $W^*_{2,2}$ which is essentially our (22·2). Engel (1902), following up Study's work directly, broadened the fundamental ideas, but does not seem to have added largely to the results relevant to our present purpose.

Gherardelli (1941) gave an exceedingly elegant study of $W^*_{2,2}$ by purely geometrical methods, scarcely making any use of co-ordinates, and building up its base and intersection properties from the relation which will appear in our notation as (25·1); this he obtained from the Plücker relation between the order and class of a plane curve, and the number of its cusps and inflexions; and he obtained the same model as Study did, as projective model of the simple linear system of minimum grade.

Meanwhile Severi (1940) had generalized the problem somewhat, following up an approach already indicated by Study, which consists of regarding a second-order element (or sequence $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$) as arising by the colaescence of two first-order elements, in the same way as we naturally regard the first-order element $\mathsf{P}_0\,\mathsf{P}_1$ as arising by the coalescence of two explicit points. This leads to the mapping of the second-order elements of the plane on certain first-order elements of $W^*_{2,1}$, namely those at each point of $W^*_{2,1}$ whose tangents are coplanar with the two generating lines of $W^*_{2,1}$ through the point. This would presumably generalize to the mapping of the sequences $\mathsf{P}_0 \ldots \mathsf{P}_n$ of S_r on certain sequences $\mathsf{P}_k \ldots \mathsf{P}_n$ on $W^*_{r,k}$, namely, those at each point of $W^*_{r,k}$ that are on a certain algebroid sheet of

r dimensions defined at the point; but promising as this idea looks, it has not for some reason proved helpful in the present investigation.

It was used by Semple (1954), however, in a further broad study of the whole problem, which contained amongst other things the first substantial results for $r > 2$, namely a parametrization of $W_{3,2}$ that is, except for notation, identical with our (14·1), as well as a detailed geometrical study of this variety. Finally, Longo (1955) gave a complete description of $W^*_{r,2}$ for all r, constructing the loci which we call Ψ^r_2, $\Psi^r_{\bar{2}}$ as images of the partial flag manifolds representing point-line-plane and point-line combinations, respectively, and generating $W^*_{r,2}$ by S_{r-1}'s joining a point of the latter to a corresponding S_{r-2} of the former. His model is just what is given by our (27·2) for $r = 3$, and its obvious generalization for higher values of r. Quite recently the base and intersection theory on Longo's model of $W^*_{r,2}$ has been studied in detail by Zobel (1960) in terms of the Schubert conditions on the origin, tangent, and osculating plane.

In nearly 60 years however no significant progress had been made with the problem for $n > 2$. At a discussion of the whole subject in a recent seminar in London, the view was expressed that (on account of the increasing variety of types of singular branch with increasing n) any unexceptionally one-one model of the sequences $\mathsf{P}_0 \ldots \mathsf{P}_n$ $(n \geqslant 3)$ might well prove to have singularities, or even to be reducible; and on my expressing some confidence to the contrary, I was challenged to produce a detailed description of $W_{3,3}$. To this challenge the present paper is intended as the answer; as an obvious preliminary, however, I have made a fairly detailed study of $W_{2,3}$, which is published elsewhere (Du Val 1961), but some of the results of which will naturally be included in their appropriate place in the sequel.

PART I. $W_{2,n}$

3. The simple branch in the plane

First of all then, taking affine co-ordinates (x, y) in a plane S_2, with origin at a fixed point P_0, we consider a simple algebroid branch at P_0 with the generic point

$$x = a_1 t + a_2 t^2 + a_3 t^3 + \ldots, \quad y = b_1 t + b_2 t^2 + b_3 t^3 + \ldots, \tag{3·1}$$

where t is an indeterminate. If the coefficients $a_1, b_1, \ldots$ are in the ground field this is a particular branch; if they are themselves indeterminate it is a generic branch. We assume in any case for the present that a_1, b_1 are not both zero, so that the branch is simple, and the sequence of points $\mathsf{P}_0 \mathsf{P}_1 \ldots$ on it is free. It is familiar that this sequence is determined as far as P_n by the coefficients $a_1, b_1, \ldots, a_n, b_n$; conversely, every simple branch through $\mathsf{P}_0 \ldots \mathsf{P}_n$ has a parametrization of type (3·1) reducible, by a regular transformation

$$t \to k_1 t + k_2 t^2 + k_3 t^3 + \ldots \quad (k_1 \neq 0) \tag{3·2}$$

to a form in which the coefficients are actually the same as in (3·1) as far as a_n, b_n. We are therefore interested in functions of these coefficients which are t-invariant, i.e. which on making the substitutions

$$a_1 \to k_1 a_1, \quad a_2 \to k_1^2 a_2 + k_2 a_1, \quad a_3 \to k_1^3 a_3 + 2k_1 k_2 a_2 + k_3 a_1, \quad \ldots$$

induced by (3·2) are merely multiplied by a power of k_1, say k_1^s: the exponent s we shall as usual call the weight of the t-invariant, and by its rank we shall mean the highest ordinal suffix n of the coefficients $a_1, b_1, \ldots, a_n, b_n$ of which it is a function, i.e. of the implicit points $\mathsf{P}_1, \ldots, \mathsf{P}_n$ on which it depends.

We have also of course to consider the invariant behaviour of these functions under a general linear transformation on (x, y). The expressions

$$D_{ij} = a_i b_j - a_j b_i$$

are of course (x, y)-invariants, though (for the simple branch now in question) only D_{12} is t-invariant. The t-invariants we shall obtain in the first instance turn out to be polynomials which we shall call (a, D) forms, homogeneous (say of degree h) in $a_1 \ldots a_n$ and also homogeneous (say of degree k) in $D_{12} \ldots D_{1n}$. (The degree h is defined without regard to the entry of $a_1, \ldots, a_n$ into $D_{12}, \ldots, D_{1n}$, so that the form is actually of degree $h+k$ in $a_1 \ldots a_n$ and k in $b_1 \ldots b_n$.) A general linear (affine) transformation

$$x \to px + qy, \quad y \to rx + sy \quad (ps - qr \neq 0) \tag{3·3}$$

induces on such a form $\mathbf{F}$ $(= \mathbf{F}_0)$ the transformation

$$\mathbf{F} \to (ps - qr)^k \sum_{i=0}^{h} p^{h-i} q^i \mathbf{F}_i,$$

where $\mathbf{F}_i$ is the form of degrees $h-i$ in $(a_1, \ldots, a_n)$ and i in $(b_1, \ldots, b_n)$ arising from $\mathbf{F}$ by the ordinary polarization process. (Note however that to avoid fractional coefficients which would otherwise occur we have defined $\mathbf{F}_i$ as the coefficient of $(ps-qr)^k p^{h-i} q^i$, not of $\binom{h}{i}(ps-qr)^k p^{h-i} q^i$, which would accord better with the usual notation for polars, and also for tensors.) The transformation (3·3) in fact induces on $\mathbf{F}_0, \ldots, \mathbf{F}_h$ a linear transformation precisely similar, except for the factor $(ps-qr)^k$, to that induced on the monomials $x^h, x^{h-1}y, \ldots, y^h$. $\mathbf{F}_1, \ldots, \mathbf{F}_h$ will be called the tensor companions of $\mathbf{F}$. Any polynomial which is either an (a, D) form or a tensor companion of one will be called an (a, b, D) form.

4. The basic t-invariants

To form t-invariants of higher rank from those of lower rank, we apply to the branch (1·1) the standard dilating transformation

$$x^{(1)} = x, \quad y^{(1)} = \frac{y}{x} - \frac{b_1}{a_1}, \tag{4·1}$$

which maps the first neighbourhood of P_0 on the line $x^{(1)} = 0$ and brings the explicit origin $\mathsf{P}_1^{(1)}$ of the transformed branch to the origin of the co-ordinate system $(x^{(1)}, y^{(1)})$. The parametric equations of the transformed branch are

$$x^{(1)} = a_1 t + a_2 t^2 + a_3 t^3 + \ldots, \quad y^{(1)} = d_1 t + d_2 t^2 + d_3 t^3 + \ldots, \tag{4·2}$$

where

$$d_1 = \frac{1}{a_1^2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad d_2 = \frac{1}{a_1^3}\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_2 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad d_3 = \frac{1}{a_1^4}\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}, \quad \ldots$$

and we note that the numerators of these expressions are (a, D) forms, which can be written

$$\left.\begin{aligned}
a_1^2 d_1 &= D_{12},\\
a_1^3 d_2 &= -a_2 D_{12} + a_1 D_{13},\\
a_1^4 d_3 &= (a_2^2 - a_1 a_3) D_{12} - a_1 a_2 D_{13} + a_1^2 D_{14},\\
a_1^5 d_4 &= (2a_1 a_2 a_3 - a_1^2 a_4 - a_2^3) D_{12} + a_1(a_2^2 - a_1 a_3) D_{13} - a_1^2 a_2 D_{14} + a_1^3 D_{15},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{4·3}$$

It is immediately obvious that if $\mathbf{F}(a, b)$ is any polynomial t-invariant of weight s and of rank n, the result of substituting $d_1, \ldots, d_n$ for $b_1, \ldots, b_n$, and then putting for $d_1, \ldots, d_n$ their values from (4·3), is a fraction whose denominator is a power of a_1, say a_1^q, and whose numerator is a t-invariant polynomial of weight $s+q$ and of rank $n+1$, since this fraction is the corresponding t-invariant of the transformed branch. This process we shall call the dilating substitution. In particular, if $\mathbf{F}$ is an (a, D) form, the dilating substitution consists simply in substituting for each D_{ij}, $\Delta_{ij} = a_i d_j - a_j d_i$; and we have at once from (4·3)

$$\left.\begin{aligned}
\Delta_{12} &= (-2a D_{12} + a_1 D_{13})/a_1^2,\\
\Delta_{13} &= \{(a_2^2 - 2a_1 a_3) D_{12} - a_1 a_2 D_{13} + a_1^2 D_{14}\}/a_1^3,\\
\Delta_{14} &= \{(2a_1 a_2 a_3 - 2a_1^2 a_4 + a_2^3) D_{12} + a_1(a_2^2 - a_1 a_3) D_{13} - a_1^2 a_2 D_{14} + a_1^3 D_{15}\}/a_1^4,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{4·4}$$

Since now $\mathbf{a} = a_1$, $\mathbf{b} = b_1$, are t-invariants of weight 1 and rank 1, $\mathbf{H}^{(2)} = \mathbf{D} = D_{12}$, being the numerator in the expression for d_1, is a t-invariant of weight 3 and of rank 2. Hence again, the numerator in the expression for Δ_{12}, namely

$$\mathbf{H}^{(3)} = \mathbf{G} = -2a_2 D_{12} + a_1 D_{13}$$

is a t-invariant of weight 5 and of rank 3. Making again the dilating substitution on this we see that

$$-2a_2 \Delta_{12} + a_1 \Delta_{13} = \mathbf{H}^{(4)}/a_1^2 = \mathbf{I}/a_1^2,$$

where
$$\mathbf{H}^{(4)} = \mathbf{I} = (5a_2^2 - 2a_1 a_3) D_{12} - 3a_1 a_2 D_{13} + a_1^2 D_{14}$$

is a t-invariant of weight 7 and of rank 4; similarly

$$(5a_2^2 - 2a_1 a_3) \Delta_{12} - 3a_1 a_2 \Delta_{13} + a_1^2 \Delta_{14} = \frac{\mathbf{H}^{(5)}}{a_1^2} = \frac{\mathbf{L}}{a_1^2},$$

where $\mathbf{H}^{(5)} = \mathbf{L} = (12a_1 a_2 a_3 - 2a_1^2 a_4 - 14a_2^3) D_{12} + 3a_1(3a_2^2 - a_1 a_3) D_{13} - 4a_1^2 a_2 D_{14} + a_1^3 D_{15}$,

is a t-invariant of weight 9 and rank 5; and so on.

It is obvious that proceeding in this way we obtain a sequence of t-invariants $\mathbf{H}^{(n)}$ of rank n and of weight $2n-1$, which are (a, D) forms, linear in $D_{12}, \ldots, D_{1n}$, and of degree $n-2$ in $a_1, \ldots, a_{n-1}$, such that under the dilating substitution

$$\mathbf{H}^{(n)} \to \mathbf{H}^{(n+1)}/a_1^2 \quad (n = 2, 3, \ldots)$$

we note also that the coefficient of D_{1n} in $\mathbf{H}^{(n)}$ is a_1^{n-2}, and that of each D_{1j} is divisible by a_1^{j-2}. We shall call $\mathbf{H}^{(n)}$ the basic t-invariant of rank n of the branch (3·1).

We now define the parameters

$$\mu_1 = b_1/a_1, \quad \mu_n = \mathbf{H}^{(n)}/a_1^{2n-1} \quad (n = 2, 3, \ldots),$$

and we remark that μ_n is the result of the dilating substitution on μ_{n-1}. Every free sequence $P_0, \ldots, P_n$ for which $a_1 \neq 0$, i.e. such that the line P_0P_1 is not the y axis, defines uniquely the values of $\mu_1, \ldots, \mu_n$; conversely, every set of values of $\mu_1, \ldots, \mu_n$ defines uniquely a free sequence $P_0, \ldots, P_n$; this is in fact true for $n = 1$, since μ_1 is the gradient of the tangent of the branch (3·1); and if we assume it for any value of n, we see that the values of $\mu_2, \ldots, \mu_{n+1}$ define uniquely the sequence $P_1^{(1)}, \ldots, P_{n+1}^{(1)}$ on the transformed branch (4·2), so that the values of $\mu_1, \ldots, \mu_{n+1}$ define uniquely the points $P_0, \ldots, P_{n+1}$ on (3·1).

There is of course nothing particularly new about these basic t-invariants, though perhaps something new in the point of view from which they are here approached. If we choose the parameter t (by a suitable transformation (3·2)) so that $a_1 = 1$, $a_i = 0$ (for all $i \geqslant 2$) we see that $\mathbf{H}^{(n)} = b_n$, so that the absolute t-invariants $\mu_1, \mu_2, \ldots$ are simply the coefficients in the expansion of y as a power series in x; and the polynomials $\mathbf{H}^{(n)}$ are simply the numerators in the expressions for the successive derivatives of y with respect to x, in terms of those of x, y with respect to t (apart from some differences in the numerical coefficients, due to the suppression of the factorial denominators in the expansions used, regarded as Taylor series.)

If now $\mathbf{F}$, $\mathbf{F}'$ are any two t-invariant polynomials of the same weight s, and of rank $\leqslant n$, the ratio $\mathbf{F}/\mathbf{F}'$ is uniquely determined by the sequence $P_0, \ldots, P_n$, and hence is a rational function of $\mu_1, \ldots, \mu_n$. If $\mathbf{F}' = \mathbf{a}^s$, since for any values of $(a_1, b_1, \ldots, a_n, b_n)$ in K (or any algebraic extension K' of K) only provided $a_1 \neq 0$, $\mathbf{F}/\mathbf{a}^s$ has a well-defined value in K (or in K'), it has one for any values of $\mu_1, \ldots, \mu_n$ in K (or in K'), so that the rational function

$$\mathbf{F}/\mathbf{a}^s = \phi(\mu_1, \ldots, \mu_n)$$

is a polynomial. This means that there is an exponent m such that $\mathbf{a}^m\mathbf{F}$ is a polynomial in $(\mathbf{a}, \mathbf{b}, \mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n)})$. In particular if $\mathbf{F}$ is an (a, D) form of degrees h, k in $(a_1, \ldots)$ and $(D_{12}, \ldots)$, this polynomial must be homogeneous of degree k in $(\mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n)})$, since these are linear in $(D_{12}, \ldots, D_{1n})$, and free from $\mathbf{b}$; moreover, it is easily calculated that the exponent of $\mathbf{a}$ in every term must be $3k+2h-s$. Thus if $\mathbf{F}$ is not divisible by $\mathbf{a}$, it is a form in $(\mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n)})$ only, without $\mathbf{a}$ or $\mathbf{b}$.

In this case the tensor companions of $\mathbf{F}$ are forms of the same degree k in $(\mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n)})$ and their tensor companions; for applying to the relation

$$\mathbf{F} = \psi(\mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n)})$$

the linear transformation (3·3), with $q/p = \sigma$, $ps - qr = 1$, we have

$$\mathbf{F} + \sigma\mathbf{F}_1 + \ldots + \sigma^h\mathbf{F}_h = \psi(\mathbf{H}^{(2)}, \mathbf{H}^{(3)} + \sigma\mathbf{H}_1^{(3)}, \ldots, \mathbf{H}^{(n)} + \ldots + \sigma^{n-2}\mathbf{H}_{n-2}^{(n)}) \tag{4·5}$$

identically in σ; and equating coefficients of like powers of σ, we obtain expressions of the required form for $\mathbf{F}_1, \ldots, \mathbf{F}_h$.

5. Further principal t-invariants

We now define inductively certain further t-invariants which we shall require, and which we call the principal t-invariants, including in this term the basic t-invariants. $\mathbf{a}$, $\mathbf{b}$ are the only principal t-invariants of rank 1; and given all those of rank n, those of rank $n+1$ are the (a, D) forms arising as numerators on applying the dilating transformation to all those of rank n, together with the tensor companions of these (a, D) forms.

The number of these increases rather rapidly with the rank, and a complete notation for them requires some symbols with $n-2$ ordinal indices for those of rank n; before introducing this, to illustrate the general build up of the scheme, we shall find, and express as forms in the basic t-invariants, all those of rank $\leqslant 5$. It is for this purpose that we have given the alternative names $\mathbf{D}$, $\mathbf{G}$, $\mathbf{I}$, $\mathbf{L}$ to $\mathbf{H}^{(2)}$, $\mathbf{H}^{(3)}$, $\mathbf{H}^{(4)}$, $\mathbf{H}^{(5)}$; and we shall use separate single letters for all the principal t-invariant (a, D) forms of rank $\leqslant 5$, reserving a single ordinal suffix to distinguish their tensor companions.

We note first that as a_n, b_n enter into $\mathbf{H}^{(n)}$ only in the term $a_1^{n-2}D_{1n}$, and into $\mathbf{H}_1^{(n)}$ only in the term $(n-2)\,a_1^{n-3}b_1D_{1n}$, $\mathbf{aH}_1^{(n)}-(n-2)\,\mathbf{bH}^{(n)}$ does not contain a_n, b_n at all, and is thus of rank $n-1$. It is in fact an (a, D) form, quadratic in $(D_{12}, \ldots, D_{1,n-1})$, and is most simply obtained by sutstituting $0, D_{12}, \ldots, D_{1,n-1}$ for $b_1, b_2, \ldots, b_{n-1}$ in the expression for $\mathbf{H}_1^{(n)}$ as an (a, b, D) form. It is thus a quadratic form in $\mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n-1)}$, and its weight shows that it is a linear combination of the products $\mathbf{H}^{(i+1)}\mathbf{H}^{(n-i)}$, for $1 \leqslant i \leqslant \frac{1}{2}(n-1)$. The tensor companions of this (a, D) form are the similar differences $i\mathbf{aH}_i^{(n)}-(n-i-1)\,\mathbf{bH}_{i-1}^{(n)}$. Thus from

$$\mathbf{G} = -2a_2D_{12}+a_1D_{13}, \quad \mathbf{G}_1 = -2b_2D_{12}+b_1D_{13}$$

we have
$$\mathbf{aG}_1-\mathbf{bG} = -2\mathbf{D}^2; \tag{5·1}$$

similarly from

$$\begin{aligned}
\mathbf{I} &= (-2a_1a_3+5a_2^2)\,D_{12}-3a_1a_2D_{13}+a_1^2D_{14},\\
\mathbf{I}_1 &= (-2a_1b_3+10a_2b_2-2a_3b_1)\,D_{12}-3(a_1b_2+a_2b_1)\,D_{13}+a_1b_1D_{14},\\
\mathbf{I}_2 &= (-2b_1b_3+5b_2^2)\,D_{12}-3b_1b_1^2D_{13}+b_1D_{14}
\end{aligned}$$

we have
$$\mathbf{aI}_1-2\mathbf{bI} = -5\mathbf{DG}, \quad 2\mathbf{aI}_2-\mathbf{bI}_1 = -5\mathbf{DG}_1; \tag{5·2}$$

and similarly at the next stage

$$\left.\begin{aligned}
\mathbf{aL}_1-3\mathbf{bL} &= -6\mathbf{DI}-3\mathbf{G}^2,\\
2\mathbf{aL}_2-2\mathbf{bL}_1 &= -6\mathbf{DI}_1-6\mathbf{GG}_1,\\
3\mathbf{aL}_3-\mathbf{bL}_2 &= -6\mathbf{DI}_2-3\mathbf{G}_1^2,
\end{aligned}\right\} \tag{5·3}$$

and so on.

We can now express all the principal t-invariant (a, D) forms as forms in the basic t-invariants. Thus applying the dilating substitution to (5·1), as

$$\mathbf{bG}-2\mathbf{D}^2 \to \frac{\mathbf{D}}{\mathbf{a}^2}\frac{\mathbf{I}}{\mathbf{a}^2}-2\left(\frac{\mathbf{G}}{\mathbf{a}^2}\right)^2,$$

we see that
$$\mathbf{G}_1 \to \mathbf{J}/\mathbf{a}^5,$$

where
$$\begin{aligned}
\mathbf{J} &= \mathbf{DI}-2\mathbf{G}^2\\
&= -(2a_1a_3+3a_2^2)\,D_{12}^2+5a_1a_2D_{12}D_{13}+a_1^2(D_{12}D_{14}-2D_{13}^2)
\end{aligned} \tag{5·4}$$

and its tensor companions
$$\left.\begin{aligned}
\mathbf{J}_1 &= \mathbf{DI}_1-4\mathbf{GG}_1,\\
\mathbf{J}_2 &= \mathbf{DI}_2-2\mathbf{G}_1^2
\end{aligned}\right\} \tag{5·5}$$

are t-invariants of rank 4 and of weight 10. These satisfy, analogously to (5·2),

$$\left.\begin{aligned}
\mathbf{aJ}_1-2\mathbf{bJ} &= \mathbf{D}(\mathbf{aI}_1-2\mathbf{bI})-4\mathbf{G}(\mathbf{aG}_1-\mathbf{bG})\\
&= 3\mathbf{D}^2\mathbf{G},\\
2\mathbf{aJ}_2-\mathbf{bJ}_1 &= 3\mathbf{D}^2\mathbf{G}_1.
\end{aligned}\right\} \tag{5·6}$$

$\mathbf{I}$. $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{J}$, $\mathbf{J}_1$, $\mathbf{J}_2$ are the principal t-invariants of rank 4.

Similarly, applying the dilating substitution to (5·2) we find that

$$\mathbf{I}_1 \to \mathbf{M}/\mathbf{a}^5, \quad \mathbf{I}_2 \to \mathbf{N}/\mathbf{a}^8,$$

$$\mathbf{M} = 2\mathbf{DL}-5\mathbf{GI},$$
$$\mathbf{N} = \tfrac{1}{2}(\mathbf{DM}-5\mathbf{GJ}) = \mathbf{D}^2\mathbf{L}-5\mathbf{DGI}+5\mathbf{G}^3$$

are t-invariant (a, D) forms of rank 5 and of weights 12, 15; and applying it to (5·4), (5·5) we find that

$$\mathbf{J} \to \mathbf{P}/\mathbf{a}^4, \quad \mathbf{J}_1 \to \mathbf{Q}/\mathbf{a}^7, \quad \mathbf{J}_2 \to \mathbf{R}/\mathbf{a}^{10},$$

where

$$\mathbf{P} = \mathbf{GL}-2\mathbf{I}^2,$$
$$\mathbf{Q} = \mathbf{GM}-4\mathbf{IJ} = 2\mathbf{DGL}-4\mathbf{DI}^2+3\mathbf{G}^2\mathbf{I},$$
$$\mathbf{R} = \mathbf{GN}-2\mathbf{J}^2 = \mathbf{D}^2\mathbf{GL}-2\mathbf{D}^2\mathbf{I}^2+3\mathbf{DG}^2\mathbf{I}-3\mathbf{G}^4$$

are t-invariant (a, D) forms of rank 5 and of weights 14, 17, 20. As $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$ are cubic and $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ quartic in $(a_1, \ldots, a_4)$, these with their tensor companions are 27 in number, and are all the principal t-invariants of rank 5.

Clearly we can go on like this indefinitely. We can denote each principal t-invariant of rank n by a symbol $(j_1, \ldots, j_{n-2})$ consisting of $n-2$ integers, by the rule that the basic t-invariant $\mathbf{H}^{(n)}$ is $(0, \ldots, 0)$, the (a, D) form arising as numerator on applying the dilating substitution to $(j_1, \ldots, j_{n-3})$ is $(j_1, \ldots, j_{n-3}, 0)$, and the values of j_{n-2} distinguish the tensor companions of this; thus $\mathbf{G}$, $\mathbf{G}_1$ are (0), (1); $\mathbf{I}, \mathbf{I}_1, \mathbf{I}_2, \mathbf{J}, \mathbf{J}_1, \mathbf{J}_2$ are $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$; and $\mathbf{L}, \ldots \mathbf{L}_3, \mathbf{M}, \ldots, \mathbf{M}_3, \mathbf{N}, \ldots, \mathbf{N}_3, \mathbf{P}, \ldots, \mathbf{P}_4, \mathbf{Q}, \ldots, \mathbf{Q}_4, \mathbf{R}, \ldots, \mathbf{R}_4$ are $(0, 0, 0)$, $\ldots$, $(0, 0, 3)$, $(0, 1, 0)$, $\ldots$, $(0, 1, 3)$, $(0, 2, 0)$, $\ldots$, $(0, 2, 3)$, $(1, 0, 0)$, $\ldots$, $(1, 0, 4)$, $(1, 1, 0)$, $\ldots$, $(1, 1, 4)$, $(1, 2, 0)$, $\ldots$, $(1, 2, 4)$; and so on. We can find the degrees and weights of these, and the limits of the various j_i's, by remarking that the (a, D) form $(j_1, \ldots, j_{n-3}, 0)$, when expressed as a form in the basic t-invariants, contains a term

$$\mathbf{D}^{j_{n-3}}\mathbf{G}^{j_{n-4}} \ldots (\mathbf{H}^{(n-2)})^{j_1}\mathbf{H}^{(n)}$$

with some numerical coefficient whose value need not concern us; this is seen to be the case for $n \leqslant 5$, from the explicit expressions we have found; and if we assume it for any value of n, the application of the dilating substitution to $(j_1, \ldots, j_{n-3}, 0)$ gives the result directly for $(j_1, \ldots, j_{n-3}, 0, 0)$, and its application to the relations analogous to (5·6) (and similarly obtained) which express

$$j_{n-2}\mathbf{a}(j_1, \ldots, j_{n-2}) - (h-j_{n-2}+1)\,\mathbf{b}(j_1, \ldots, j_{n-2}-1)$$

as a t-invariant shows that $(j_1, \ldots, j_{n-2}, 0)$ has an expression containing a term

$$\frac{h-j_{n-2}+1}{j_{n-2}}\mathbf{D}(j_1, \ldots, j_{n-2}-1, 0).$$

We see accordingly that $(j_1, \ldots, j_{n-2})$ is a form of degree k in $(\mathbf{H}^{(2)}, \ldots, \mathbf{H}^{(n)})$ and their tensor companions (the latter only if $j_{n-2} \geqslant 1$), and is an (a, b, D) form of degrees k in $(D_{12}, \ldots, D_{1n})$, $h-j_{n-2}$ in $(a_1, \ldots, a_{n-1})$, and j_{n-2} in $(b_1, \ldots, b_{n-1})$, and of weight s, where

$$k = 1+j_{n-3}+j_{n-4}+\ldots+j_1,$$
$$h = n-2+j_{n-4}+2j_{n-5}+\ldots+(n-4)\,j_1,$$
$$s = 2n-1+3j_{n-3}+5j_{n-4}+\ldots+(2n-5)\,j_1;$$

and since the range of values of j_{n-2} is from 0 to h,

$$0 \leqslant j_i \leqslant i + j_{i-2} + 2j_{i-3} + \ldots + (i-2)\, j_1 \quad (i = 3, \ldots, n-2).$$

From this last it follows that the maximum values of $(j_1, \ldots, j_{n-2})$ are $(1, 2, 4, \ldots, 2^{n-3})$; thus $\mathbf{G}_1$, $\mathbf{J}_2$, $\mathbf{R}_4$ are (1), (1, 2), (1, 2, 4), and so on. We notice also in passing that

$$s = 2h + 3k. \tag{5·7}$$

6. t-Invariants of a singular branch

If the branch parametrized in (3·1) is not simple $a_1 = b_1 = 0$, $D_{1j} = 0$ for all j, and all the invariants defined in the last two sections vanish. Nevertheless such a branch has t-invariants of its own, expressible as (a, D) or (a, b, D) forms in the coefficients in the expansion (3·1); and the principal t-invariants form a well-defined set in this case also, though for a given rank we shall find they are less numerous than those of a simple branch.

Before defining these, however, we just say a word on the generic branch of a given multiplicity type. The type of a branch is specified by the multiplicities on it of the points $\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \ldots$, say $m, m_1, m_2, \ldots$, where

$$m_i = \sum_{j=i+1}^{i+k_i} m_j, \tag{6·1}$$

$\mathsf{P}_{i+1}, \ldots, \mathsf{P}_{i+k_i}$ being the points of the sequence that are proximate to P_1. Only a finite number of $m, m_1, \ldots$ are different from unity, and we can denote the type by $\{m, m_1, \ldots\}$, the sequence being carried as far as the first 1 in it. In particular m is called the order of the branch. For every unfree sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ there is a unique multiplicity type of minimum order for branches passing through all the points of the sequence, obtained by putting $m_n = 1$, and determining $m_{n-1}, m_{n-2}, \ldots$ in turn from (6·1); and the actual multiplicity type of any branch is that of minimum order for the sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ on it, provided n is chosen high enough for the sequence to include all the unfree points of the branch.

A branch of order m has a parametrization (3·1) with $a_i = b_i = 0$ $(i = 1, \ldots, m-1)$, but at least one of $a_m, b_m \neq 0$. The tangent is

$$y = (b_m/a_m)\, x \tag{6·2}$$

and if this meets the branch in $m+m'$ coincident points $D_{ij} = 0$ for $i < j < m+m'$, but $D_{m(m+m')} \neq 0$. (m' is called the class of the branch, and is equal to $\sum_{i=1}^{k} m_i$, where $\mathsf{P}_0 \mathsf{P}_1 \ldots \mathsf{P}_k$ are the points of the sequence consecutive on the tangent.) If $a_m \neq 0$ the dilating transformation takes the form

$$x^{(1)} = x, \quad y^{(1)} = \frac{y}{x} - \frac{b_m}{a_m}, \tag{6·3}$$

mapping as before the first neighbourhood of P_0 on the line $x^{(1)} = 0$, and bringing the origin $\mathsf{P}_1^{(1)}$ of the transformed branch to the origin of the new co-ordinate system. The transformed branch has now a parametrization (4·2), in which however $d_1, d_2, \ldots$ are given not by (4·3), but by precisely similar expressions with all the suffixes of the a's and D's augmented by $m-1$.

By the generic branch of a given multiplicity type we shall mean one with a parametrization (3·1), in which the coefficients are indeterminates, subject only to certain algebraic relations, which we shall call the type equations, which are necessary and sufficient to ensure that the branch is of the type in question. Amongst these are

$$a_i = b_i = 0 \quad (i < m), \quad D_{ij} = 0 \quad (i < j < m+m_1), \tag{6·4}$$

and there may be others, consisting of the vanishing of certain (a, D) forms, together of course with all their tensor companions, since a generic branch of given type is still a generic branch of that type after an affine transformation (3·3). If m, m_1 are mutually prime, or if $m = m_1$, $m_2 = 1$, (6·4) are sufficient.

To obtain these type equations, and to define the principal t-invariants of the generic branch of any type, we first define the species. A branch, and the sequence $\mathsf{P}_0 \mathsf{P}_1 \ldots$ of points on it, are of species i, if P_{i-1} is the last point of the sequence to which any other point of the sequence is indirectly proximate. If the branch is simple, i.e. if the sequence is free, containing no indirectly proximate points, both are of species 0. It is clear that when we apply the dilating transformation (6·3) to a branch of species $i \geqslant 1$, the transformed branch is of species $i-1$. It is also clear that there are only a finite number of proximity types of sequence $\mathsf{P}_0, \ldots, \mathsf{P}_n$ of any given species, for any given n.

If then we assume that we know all the type equations for all types of branch of species $i-1$, to find those of any given type of species i, we have only to apply the transformation (6·3), and substitute the coefficients in the expansion (4·2), expressed as fractions whose numerators are (a, D) forms and whose denominators are powers of a_m, for the coefficients a_i, b_i in each of the type equations for the transformed branch, which are all known, since this branch is of species $i-1$; and we obtain directly those of the type equations for the given branch that consist in the vanishing of (a, D) forms; the rest are the tensor companions of these.

We may illustrate this by finding the type equations for the generic branch of type $\{2, 2, 2, 1, \ldots\}$, which is the simplest for which (6·4) are not sufficient. (6·4) becomes in this case $a_1 = b_1 = D_{23} = 0$, so that for the transformed branch

$$d_1 = 0, \quad d_2 = \frac{D_{24}}{a_2^2}, \quad d_3 = \frac{-a_3 D_{24} + a_2 D_{25}}{a_2^3}, \quad \ldots; \tag{6·5}$$

and as the transformed branch is of type $2, 2, 1, \ldots$ these must satisfy $a_1 = d_1 = \Delta_{23} = 0$ (where as before $\Delta_{ij} = a_i d_j - a_j d_i$), giving the further relation $a_2 D_{25} - 2a_3 D_{24} = 0$ for the given branch, and hence also of course $b_2 D_{25} - 2b_3 D_{24} = 0$.

We now, by a similar inductive process over the species, define the principal t-invariants of the generic branch of any type, those for a simple branch, the only type of species 0, having been defined already. We assume therefore that we have defined as (a, b, D) forms a complete set of principal t-invariants of each rank $1, 2, \ldots$ for every type of branch of species $i-1$; and consider a generic branch of some type of species i. If its order is m, a_m, b_m are obviously t-invariants of weight m, and we define their rank to be unity, since they determine, and (except for the common weight factor) are determined by the point P_1. Making now the transformation (6·3) the transformed branch is of species $i-1$, and by hypothesis its principal t-invariants are all defined, and are expressed as (a, d, Δ) forms. If $\mathbf{F}$ is any one of these, of weight s and rank n, the dilating substitution gives

$$\mathbf{F} \to \mathbf{F}^*/a_m^h$$

(for some integer h) where $\mathbf{F^*}$ is a t-invariant (a, D) form of the given branch, of weight $s+hm$; and we define its rank to be $n+1$. Actually, the concept of the rank of a t-invariant of a singular branch is a little artificial; generally the t-invariants of rank n depend only on the points $\mathsf{P}_0 \dots \mathsf{P}_n$ of the sequence; but we shall notice that when P_n is not a free point it may happen that a t-invariant of rank n depends on the next free point P_{n+k} say, all those of ranks $n+1, \dots, n+k$ being then expressible as polynomials in those of rank n. (When all the t-invariants of a given rank are polynomials in those of lower rank we shall say that there are no proper t-invariants of this rank.)

We now define the principal t-invariants of the generic branch of any type to be the t-invariants a_m, b_m of rank 1, the (a, D) forms arising in this way by the dilating substitution from all the principal t-invariants of the transformed branch, and all the tensor companions of these latter. To avoid the needless multiplication of symbols we shall, for every type of singular branch, denote by $\mathbf{a}'$, $\mathbf{b}'$ the t-invariants a_m, b_m of rank 1 (whatever the value of m); and similarly $\mathbf{D}'$ will denote $D_{m(m+m_1)}$, the first of the D_{ij}'s whose vanishing is not one of the type equations; this is always the only principal t-invariant of rank 2, arising from the invariant $\mathbf{b}'$ or $\mathbf{b}$ of the transformed branch.

We will illustrate this by finding the principal t-invariants of the first few ranks for four types of branch, which are those that have to be considered in parametrizing $W_{2,3}$:

I. Ordinary cuspidal branch, order 2, species 1, type $\{2, 1, \dots\}$, proximity type 2, type equations $a_1 = b_1 = 0$;

$$d_1 = D_{23}/a_2^2, \quad d_2 = (-a_3 D_{23} - a_2 D_{24})/a_2^3, \quad \dots$$

and since $a_1 = 0$ $\qquad \Delta_{1j} = -a_j d_1 = -a_j D_{23}/a_2^2 \quad (j = 2, 3, \dots);$

putting these values of d_i, Δ_{1j} in place of b_i, D_{1j} in the principal t-invariants of a simple branch we have

$$\mathbf{b} \to \frac{\mathbf{D}'}{\mathbf{a}'^2}, \quad \mathbf{D} \to -\frac{\mathbf{D}'}{\mathbf{a}'}, \quad \mathbf{G} \to -\mathbf{D}', \quad \mathbf{G}_1 \to \frac{\mathbf{D}'\mathbf{G}'}{\mathbf{a}'^4}, \quad \dots,$$

where $\qquad \mathbf{G}_1' = -3a_3 D_{23} + 2a_2 D_{24}$

and its tensor companion $\qquad \mathbf{G}_1 = -3b_3 D_{23} + 2b_2 D_{24}$

are the only proper principal t-invariants of rank 4, and there are none of rank 3.

II. Rhamphoid cuspidal branch, order 2, species 2, type $\{2, 2, 1, \dots\}$, proximity type 3; $a_1 = b_1 = D_{23} = 0$; $d_1, d_2, \dots$ are given by (6·5) and consequently

$$\Delta_{1j} = 0 \quad (j = 2, 3, \dots), \qquad \Delta_{23} = (-2a_3 D_{24} + a_2 D_{25})/a_2^2,$$

so that applying the dilating substitution to the principal t-invariants of the (ordinary cuspidal) transformed branch we see that while $\mathbf{D}' = D_{24}$ is the only principal t-invariant of rank 2,

$$\mathbf{G}'' = -2a_3 D_{24} + a_2 D_{25}, \quad \mathbf{G}_1'' = -2b_3 D_{24} + b_2 D_{25}$$

are the only proper ones of rank 3, and there are none of rank 4. We remark also that $\mathbf{b}'\mathbf{G}'' - \mathbf{a}'\mathbf{G}_1'' = 0$.

III. Ordinary cubic branch, order 3, species 1, type $\{3, 1, \ldots\}$, proximity type (23); $a_1 = b_1 = a_2 = b_2 = 0$;

$$\mathbf{b} \to \mathbf{D}', \quad \mathbf{D} \to 0, \quad \mathbf{G} \to 0, \quad \mathbf{G}_1 \to -\mathbf{D}'^2/\mathbf{a}'^3, \quad \ldots$$

and while $\mathbf{D}' = D_{34}$ is the only principal t-invariant of rank 2 there are no proper t-invariants of rank 3 or 4.

IV. Cubo-quadric branch, order 3, species, 2, type $\{3, 2, 1, \ldots\}$, proximity type 2, 3; $a_1 = b_1 = a_2 = b_2 = D_{34} = 0$; the transformed branch is cuspidal, $\mathbf{D}' = D_{35}$ is the only principal t-invariant of rank 2, and again there are no proper t-invariants of rank 3 or 4.

We notice the increasing meagreness of the system of principal t-invariants as we increase the order and species of the branch, but we also verify that every principal t-invariant of the transformed branch, which is of lower species, has a perfectly definite image, so to speak, among the t-invariants of the given branch, even though this may be a function of the invariants of lower rank.

7. Parametrization of $W_{2,n}$

To see how we can obtain a parametrization of $W_{2,n}$ in terms of the t-invariants now obtained, we will begin with the trivial case $n = 1$. $W_{2,1}$ is a line, and is clearly parametrized in terms of the first rank t-invariants by putting

$$X_0 : X_1 = \mathbf{a} : \mathbf{b}.$$

Thus for any particular point P_1 in the first neighbourhood of P_0, we can parametrize the first neighbourhood of P_1, i.e. of the explicit point $\mathsf{P}_1^{(1)}$, by putting

$$X : Y = d_1 : a_1 = \frac{\mathbf{D}}{\mathbf{a}^2} : \mathbf{a} = \mathbf{aD} : \mathbf{a}^4; \tag{7·1}$$

but a linear transformation (3·3) transforms $\mathbf{aD}$ into a linear combination of $\mathbf{aD}$, $\mathbf{bD}$, and $\mathbf{a}^4$ into a linear combination of $\mathbf{a}^4$, $\mathbf{a}^3\mathbf{b}$, $\mathbf{a}^2\mathbf{b}^2$, $\mathbf{ab}^3$, $\mathbf{b}^4$; thus if we take homogeneous co-ordinates in S_6:

$$X_0 : X_1 : Y_0 : Y_1 : Y_2 : Y_3 : Y_4 = \mathbf{aD} : \mathbf{bD} : \mathbf{a}^4 : \mathbf{a}^3\mathbf{b} : \mathbf{a}^2\mathbf{b}^2 : \mathbf{ab}^3 : \mathbf{b}^4 \tag{7·2}$$

we see that a linear transformation (3·3) induces a collineation in S_6 which is a self transformation of the locus whose generic point is given by (7·2); this locus is evidently a ruled quintic surface with a directrix line $Y_i = 0$ $(i = 0, \ldots, 4)$, each generator being given by a constant ratio $\mathbf{b}/\mathbf{a}$, and being the line with equations

$$X_1 = \lambda X_0, \quad Y_i = \lambda Y_{i-1} \quad (i = 1, 2, 3, 4), \tag{7·3}$$

where $\lambda = \mathbf{b}/\mathbf{a}$ is a parameter determining the point P_1. As on each generator X_0, Y_0 are a homogeneous co-ordinate system, we see that each generator is a model of the $W_{2,1}$ which is the first neighbourhood of the corresponding P_1; we shall denote this by $W_{2,1}(\mathsf{P}_1)$. The whole ruled quintic surface is thus a model of $W_{2,2}$.

We next parametrize the second neighbourhood of each first neighbourhood point P_1, or $W_{2,2}(\mathsf{P}_1)$, by applying the dilating transformation to the monomials on the right of (7·2), obtaining in the first instance

$$\begin{aligned} &d_1\Delta : a_1\Delta : d_1^4 : d_1^3 a_1 : d_1^2 a_1^2 : d_1 a_1^3 : a_1^4 \\ &= \frac{\mathbf{G}}{\mathbf{a}^2}\frac{\mathbf{D}}{\mathbf{a}^2} : \frac{\mathbf{G}}{\mathbf{a}^2}\mathbf{a} : \left(\frac{\mathbf{D}}{\mathbf{a}^2}\right)^4 : \left(\frac{\mathbf{D}}{\mathbf{a}^2}\right)^3 \mathbf{a} : \left(\frac{\mathbf{D}}{\mathbf{a}^2}\right)^2 \mathbf{a}^2 : \left(\frac{\mathbf{D}}{\mathbf{a}^2}\right) \mathbf{a}^3 : \mathbf{a}^4 \\ &= \mathbf{a}^5\mathbf{DG} : \mathbf{a}^8\mathbf{G} : \mathbf{aD}^4 : \mathbf{a}^4\mathbf{D}^3 : \mathbf{a}^7\mathbf{D}^2 : \mathbf{a}^{10}\mathbf{D} : \mathbf{a}^{13}, \end{aligned} \tag{7·4}$$

where, as in (7·1), we have not only cleared of fractions but made unity the least exponent of $\mathbf{a}$ in any of the monomials on the right. Adjoining as before to these (a, D) form monomials all their tensor companions, we take homogeneous co-ordinates in S_{69} as follows:

$$\left.\begin{array}{ll}\left.\begin{array}{l}X_{ij} = \mathbf{a}^{5+3j-i}\mathbf{b}^i\mathbf{D}^{1-j}\mathbf{G} \\ X'_{ij} = \mathbf{a}^{5+3j-i}\mathbf{b}^i\mathbf{D}^{1-j}\mathbf{G}_1\end{array}\right\} & (j = 0, 1;\ i = 0, \ldots, 5+3j), \\ Y_{ij} = \mathbf{a}^{1+3j-i}\mathbf{b}^i\mathbf{D}^{4-j} & (j = 0, \ldots, 4;\ i = 0, \ldots, 1+3j).\end{array}\right\} \quad (7·5)$$

Because the monomials in (7·2) are isobaric, those in (7·4) and hence all those in (7·5) are isobaric (in fact, they are all of weight 13). Thus every free sequence $\mathsf{P}_0\,\mathsf{P}_1\,\mathsf{P}_2\,\mathsf{P}_3$ has a unique image or representative point in S_{69} obtained by putting into (7·5) the values of the t-invariants of any simple branch through it. As, however, by (5·1)

$$X'_{ij} = X_{i+1,j} - 2Y_{i,j+1} \quad (j = 0, 1;\ i = 0, \ldots, 4+3j) \qquad (7·6)$$

the ambient of the algebraic variety of which (7·5) gives a generic point is not actually S_{69} but S_{56}. The points of this variety for which $\mathbf{b}/\mathbf{a}$ has a particular value λ (whether generic, i.e. transcendent over the ground field, or in the latter) satisfy

$$\left.\begin{array}{ll}\left.\begin{array}{l}X_{i+1,j} = \lambda X_{ij} \\ X'_{i+1,j} = \lambda X'_{ij}\end{array}\right\} & (j = 0, 1;\ i = 0, \ldots, 4+3j), \\ Y_{i+1,j} = \lambda Y_{ij} & (j = 0, \ldots, 4;\ i = 0, \ldots, 3j)\end{array}\right\} \quad (7·7)$$

and these with (7·6) are the equations of an S_6 in which X_{0j} $(j = 0, 1)$, Y_{0j} $(j = 0, \ldots, 4)$ are independent co-ordinates; but as by (7·5) $X_{00}:X_{01}:Y_{00}:Y_{01}:Y_{02}:Y_{03}:Y_{04}$ are proportional to the right-hand member, and hence to the left-hand member, of (7·4), we see that the subspace (7·7) traces on the variety parametrized in (7·5) a ruled quintic $W_{2,2}(\mathsf{P}_1)$, which maps the second neighbourhood of the point P_1 corresponding to the particular value of λ, in exactly the same way as the ruled quintic $W_{2,2}$ parametrized in (7·2) maps the second neighbourhood of P_0 itself. Moreover, as a linear transformation (3·3) replaces each t-invariant by a linear combination of itself and its tensor companions, it induces on the variety parametrized in (7·5) a self collineation, which interchanges among themselves projectively the pencil $|W_{2,2}(\mathsf{P}_1)|$ given by the values of λ. We notice that, though the parametrization of $W_{2,2}(\mathsf{P}_1)$ does not work directly in the case $\mathbf{a} = 0$, i.e. when the tangent $\mathsf{P}_0\mathsf{P}_1$ is the y axis, the mapping of the second neighbourhood of P_1 on a ruled quintic of the pencil is precisely similar in this case to the general case because the y axis can be transformed into any other line through P_0 by a transformation (3·3). The algebraic variety whose generic point is given by (7·5) is our model of $W_{2,3}$.

We can repeat this process indefinitely. We suppose for induction that we have obtained a model of $W_{2,n-1}$ by equating the homogeneous co-ordinates in a space of suitably high dimensions to certain monomials in the principal t-invariants of rank $\leqslant n-1$, which are all of weight m_{n-1} and include all the monomials of degree m_{n-1} in $(\mathbf{a}, \mathbf{b})$. Applying the dilating transformation to these we obtain a set of monomials in the principal t-invariant (a, D) forms of rank $\leqslant n$, divided by various powers of $\mathbf{a}$, the highest such power in any denominator being in the monomial

$$\mathbf{D}^{m_{n-1}}/\mathbf{a}^{2m_{n-1}}$$

which arises from $\mathbf{b}^{m_{n-1}}$, since $\mathbf{b}$ is the only principal t-invariant that gives rise through the dilating transformation to one of more than twice its own weight. Multiplying throughout by $\mathbf{a}^{2m_{n-1}+1}$, we obtain a set of monomials all of weight m_n, and including $\mathbf{aD}^{m_{n-1}}$ and $\mathbf{a}^{m_n}$, where $m_n = 3m_{n-1}+1$, so that by induction from $m_1 = 1$ we have $m_n = \frac{1}{2}(3^n-1)$. Adjoining to these t-invariant (a, D) form monomials all their tensor companions, we have a new set of isobaric monomials in all the principal t-invariants of rank $\leqslant n$; and equating these to the homogeneous co-ordinates in space of suitably higher dimensions we obtain a generic point of an algebraic variety which we take as our model of $W_{2,n}$. Every free sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ has a unique image on this variety, because the co-ordinates are isobaric. The co-ordinates are not of course all independent but satisfy a number of linear identities (such as (7·6) in the case $n = 3$) expressing the identities such as (5·1), (5·2), (5·3), (5·6). The co-ordinate monomials which are not (a, D) forms are all expressible linearly in terms of those which are, by means of these identities and relations analogous to (7·7) relating the monomials which differ only in the exponents of $\mathbf{a}$ and $\mathbf{b}$; these last relations are the equations of a subspace tracing on $W_{2,n}$ the locus of images of sequences for which $\lambda = \mathbf{b}/\mathbf{a}$ has a particular value, i.e. in which P_1 is a particular point of the first neighbourhood of P_0; this locus we denote by $W_{2,n-1}(\mathsf{P}_1)$, and from the way in which the (a, D) form monomials were obtained from the (a, d, Δ) form monomials belonging to the transformed branch, it is clear that each $W_{2,n-1}(\mathsf{P}_1)$ maps the $(n-1)$th neighbourhood of the corresponding P_1 in exactly the same way as the $W_{2,n-1}$ from which we started maps that of P_0; $|W_{2,n-1}(\mathsf{P}_1)|$ is in fact a pencil of primals on $W_{2,n}$, each member of which is a projective image of the original $W_{2,n-1}$.

A word is perhaps not out of place as to why we multiply by a power of $\mathbf{a}$ which not only clears of fractions but leaves unity as the least exponent of $\mathbf{a}$ in all the resulting monomials. If we had merely cleared of fractions the pencil $|W_{2,n-1}(\mathsf{P}_1)|$ would have had a base point, at which all the co-ordinates vanish except that equated to $\mathbf{D}^{m_{n-1}}$, which would be a singular point of $W_{2,n}$; thus if in (7·1) we had taken the right-hand member in the form $\mathbf{D}:\mathbf{a}^3$ we should have obtained instead of the ruled quintic (7·2) the cubic cone

$$X:Y_0:Y_1:Y_2:Y_3 = \mathbf{D}:\mathbf{a}^3:\mathbf{a}^2\mathbf{b}:\mathbf{ab}^2:\mathbf{b}^3.$$

If on the other hand we multiplied by a higher power of $\mathbf{a}$, we should obtain a birational model of $W_{2,n}$, but not of minimum order; taking the right-hand member of (7·1) in the form $\mathbf{a}^m\mathbf{D}:\mathbf{a}^{m+3}$ for instance, we get a ruled surface of order $2m+3$, with minimum directrix curve of order m, which is the projective model of the complete linear system on the ruled quintic coresidual to a prime section plus $m-1$ generators.

We can illustrate the procedure by finding rapidly the parametrization of $W_{2,4}$. Applying the dilating transformation directly to (7·5) we obtain in the first instance the monomial fractions

$$\left.\begin{aligned} &\frac{\mathbf{I}}{\mathbf{a}^2}\left(\frac{\mathbf{G}}{\mathbf{a}^2}\right)^{1-j}\left(\frac{\mathbf{D}}{\mathbf{a}^2}\right)^{5+3j-i}\mathbf{a}^i \\ &\frac{\mathbf{J}}{\mathbf{a}^2}\left(\frac{\mathbf{G}}{\mathbf{a}^2}\right)^{1-j}\left(\frac{\mathbf{D}}{\mathbf{a}^2}\right)^{5+3j-i}\mathbf{a}^i \end{aligned}\right\} \quad (j = 0, 1;\ i = 0, \ldots, 5+3j),$$

$$\left(\frac{\mathbf{G}}{\mathbf{a}^2}\right)^{4-j}\left(\frac{\mathbf{D}}{\mathbf{a}^2}\right)^{1+3j-i}\mathbf{a}^i \quad (j = 0, \ldots, 4;\ i = 0, \ldots, 1+3j),$$

and multiplying by $\mathbf{a}^{27}$ and adjoining all tensor companions we obtain a set of monomials which we can equate to co-ordinates as follows:

$$\left.\begin{array}{l}
\left.\begin{array}{l}
X_{ijkl} = \mathbf{J}\mathbf{G}^{1-k-l}\mathbf{G}_1^l\mathbf{D}^{5+3k-j}\mathbf{a}^{10-4k+3j-i}\mathbf{b}^i,\\
X'_{ijkl} = \mathbf{J}_1\mathbf{G}^{1-k-l}\mathbf{G}_1^l\mathbf{D}^{5+3k-j}\mathbf{a}^{10-4k+3j-i}\mathbf{b}^i,\\
X''_{ijkl} = \mathbf{J}_2\mathbf{G}^{1-k-l}\mathbf{G}_1^l\mathbf{D}^{5+3k-j}\mathbf{a}^{10-4k+3j-i}\mathbf{b}_i,
\end{array}\right\}
\begin{array}{l}(k=0,1;\ l=0,1-k;\\ j=0,\ldots,5+3k;\\ i=0,\ldots,10-4k+3j),\end{array}\\[2ex]
\left.\begin{array}{l}
Y_{ijkl} = \mathbf{I}\mathbf{G}^{1-k-l}\mathbf{G}_1^l\mathbf{D}^{5+3k-j}\mathbf{a}^{13-4k+3j-i}\mathbf{b}^i,\\
Y'_{ijkl} = \mathbf{I}_1\mathbf{G}^{1-k-l}\mathbf{G}_1^l\mathbf{D}^{5+3k-j}\mathbf{a}^{13-4k+3j-i}\mathbf{b}^i,\\
Y''_{ijkl} = \mathbf{I}_2\mathbf{G}^{1-k-l}\mathbf{G}_1^l\mathbf{D}^{5+3k-j}\mathbf{a}^{13-4k+3j-i}\mathbf{b}_i,
\end{array}\right\}
\begin{array}{l}(k=0,1;\ l=0,1-k;\\ j=0,\ldots,5+3k;\\ i=0,\ldots,13-4k+3j),\end{array}\\[2ex]
\mathbf{Z}_{ijkl} = \mathbf{G}^{4-k-l}\mathbf{G}_1^l\mathbf{D}^{1+3k-j}\mathbf{a}^{17-4k+3j-i}\mathbf{b}^i
\quad \left\{\begin{array}{l}(k=0,\ldots,4;\ l=0,\ldots,4-k;\\ j=0,\ldots,1+3k;\\ \mathrm{i}=0,\ldots,17-4k+3j).\end{array}\right.
\end{array}\right\} \qquad (7\cdot8)$$

There are 4457 of these co-ordinates, but owing to the large number of linear identities between them the actual ambient of $W_{2,4}$ is S_{1270}.

8. The unfree sequences

The parametrization we have obtained gives us a generic point of our model of $W_{2,n}$, and a unique image on the model for every free sequence $\mathsf{P}_0, \ldots, \mathsf{P}_n$; but we have not yet proved that an unfree sequence has a well-defined image on the model, nor that every point of the model is the image of a unique sequence. We will deal with the first matter first.

The image of an unfree sequence certainly cannot be obtained, as that of any particular free sequence can, by putting into the co-ordinates of the generic point the values of a_i, b_i for some branch through the sequence; since every branch through an unfree sequence is singular, and as we have seen all the t-invariants in terms of which the generic point is defined are zero for the singular branch. Neither is there much hope of obtaining the image of the unfree sequence as a limit, even if the ground field is one (such as the real or complex number field) in which limiting processes are possible. We can of course, in such a field, let the coefficients a_i, b_i in the parametrization of a branch of one type (say a simple branch) vary continuously, and tend to limiting values for which the branch is of some more complicated type; but it does not seem to be possible to regard the sequence on the limiting form of the branch as being in any sense the limit of the sequence on the variable branch. For instance, if we vary a simple branch in this way, tending to a limit in which $a_1 = b_1 = 0$, so that the limiting form of the branch is cuspidal, the ratio b_1/a_1 may have a definite limit, and in this case it is possible to say that the point P_1 on the variable branch tends to a definite limiting position; but this is entirely unrelated to the actual point P_1 on the cuspidal branch, which depends on the limiting value of b_2/a_2 and not at all on that of b_1/a_1.

Nevertheless, we shall now prove that every unfree sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ of species i has a well-defined image point on $W_{2,n}$, whose co-ordinates can be expressed unambiguously in terms of the principal t-invariants of rank $\leqslant n$ of any branch of species i and of minimum order m passing through the sequence. This is true for $i = 0$, the case of the free sequence and the simple branch. We shall therefore assume it for all species $\leqslant i-1$, and for all n.

The dilating transformation gives us a sequence $\mathsf{P}_1^{(1)}\ldots\mathsf{P}_n^{(1)}$ of species $i-1$, lying on the transformed branch, which is likewise of species $i-1$, and minimum order for branches passing through this sequence. By the inductive hypothesis, this sequence has a well-defined image point on the appropriate member of the pencil $W_{2,n-1}(\mathsf{P}_1)$; and this point is the image of the given sequence on $W_{2,n}$. Further, in the parametrization of $W_{2,n-1}(\mathsf{P}_1)$, the image of the sequence $\mathsf{P}_1^{(1)}\ldots\mathsf{P}_n^{(1)}$ has by hypothesis co-ordinates expressible in terms of the principal t-invariants of the transformed branch, which when expressed in terms of the coefficients in the expansion of the given branch give directly those co-ordinates (in the whole ambient of $W_{2,n}$) which are equated for the generic point to (a,D) form monomials, in this case also as (a,D) form monomials; and the remaining co-ordinates are obtained from these as monomials in all the principal t-invariants of the given branch, by means of the linear identities between the co-ordinates and the equations of the ambient of $W_{2,n-1}(\mathsf{P}_1)$, using of course the value $\lambda = b_m/a_m$ appropriate to the branch of order m.

We may illustrate this process by finding explicitly the co-ordinate of the image points of all the unfree sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ on $W_{2,2}$, and those of all the unfree sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$ on $W_{2,3}$. For $n=2$ of course the problem is trivial; there is only one type of unfree sequence, type 2, on a cuspidal branch, of species 1; in place of (7·1) we have

$$X_0:Y_0 = d_1:a_1 = \mathbf{D}'/\mathbf{a}'^2:0 = \mathbf{a}':0,$$

so that by (7·3), with $\lambda = \mathbf{b}'/\mathbf{a}'$,

$$X_0:X_1 = \mathbf{a}':\mathbf{b}', \quad Y_i = 0 \quad (i = 0, \ldots, 4). \tag{8·1}$$

The locus of images of the unfree sequences on the ruled quintic $W_{2,2}$ is thus its directrix line.

For $n=3$ we have four types of unfree sequence, 2 and (23) of species 1, on branches of orders 2, 3, respectively, and 3 and 2, 3 of species 2, again on branches of orders 2, 3, respectively. For those of species 1 we have

Type 2:

$$\begin{aligned} X_{00}:X_{01}:Y_{00}:Y_{01}:Y_{02}:Y_{03}:Y_{04} &= d_1\Delta_{12}:a_1\Delta_{12}:d_1^4:d_1^3a_1:d_1^2a_1^2:d_1a_1^3:a_1^4 \\ &= -\frac{\mathbf{D}'^2}{\mathbf{a}'^3}:0:\frac{\mathbf{D}'^4}{\mathbf{a}'^8}:0:0:0:0 \\ &= -\mathbf{a}'^6:0:\mathbf{a}'\mathbf{D}'^2:0:0:0:0 \end{aligned} \tag{8·2}$$

whence from (7·6) and (7·7)

$$\left.\begin{aligned} &X_{i0} = -\mathbf{a}'^{6-i}\mathbf{b}'^i, \quad X'_{i0} = -\mathbf{a}'^{5-i}\mathbf{b}'^{i+1} \quad (i=0,\ldots,5); \\ &Y_{00} = \mathbf{a}'\mathbf{D}'^2, \quad Y_{10} = \mathbf{b}'\mathbf{D}'^2; \\ &X_{i1} = X'_{i1} = 0, \quad Y_{i1} = Y_{i2} = Y_{i3} = Y_{i4} = 0. \end{aligned}\right\} \tag{8·3}$$

Type (23):

Since $\Delta_{12} = 0$ (8·2) is proportional to

$$0:0:1:0:0:0:0$$

and again from (7·6) and (7·7)

$$\left.\begin{aligned} &Y_0 = \mathbf{a}', Y_{10} = \mathbf{b}', \\ &X_{ij} = X'_{ij} = 0, \quad Y_{i1} = Y_{i2} = Y_{i3} = Y_{i4} = 0. \end{aligned}\right\} \tag{8·4}$$

Type 3:

The least-order branch through this sequence is rhamphoid, and the transformed branch is cuspidal; applying the dilating transformation to (8·1) we have

$$\begin{aligned} X_{00}:X_{01}:Y_{00}:Y_{01}:Y_{02}:Y_{03}:Y_{04} &= d_2:a_2:0:0:0:0:0 \\ &= \frac{\mathbf{D}'}{\mathbf{a}'^2}:\mathbf{a}':0:0:0:0:0 \\ &= \mathbf{a}'^5\mathbf{D}':\mathbf{a}'^8:0:0:0:0:0 \end{aligned} \tag{8·5}$$

and again from (7·6) and (7·7)

$$\left.\begin{aligned} &X_{i0} = \mathbf{a}'^{6-i}\mathbf{b}'^{i}\mathbf{D}', \quad X'_{i0} = \mathbf{a}'^{5-i}\mathbf{b}'^{i+1}\mathbf{D}' \quad (i = 0, \ldots, 5), \\ &X_{i1} = \mathbf{a}'^{9-i}\mathbf{b}'^{i}, \quad X'_{i1} = \mathbf{a}'^{8-i}\mathbf{b}'^{i+1} \quad (i = 0, \ldots, 8), \\ &Y_{ij} = 0. \end{aligned}\right\} \tag{8·6}$$

Type 2,3:

Here again the transformed branch is cuspidal, but because the original branch is cubic we have $a_2 = 0$, and (8·5) is proportional to

$$1:0:0:0:0:0:0,$$

and from (7·6) and (7·7)

$$\left.\begin{aligned} &X_{i0} = \mathbf{a}'^{6-i}\mathbf{b}'^{i}, \quad X'_{i0} = \mathbf{a}'^{5-i}\mathbf{b}'^{i+1} \quad (i = 0, \ldots, 5), \\ &X_{i1} = X'_{i1} = Y_{ij} = 0. \end{aligned}\right\} \tag{8·7}$$

(We recall that for the quadratic branches $\mathbf{a}' = a_2$, $\mathbf{b}' = b_2$, $\mathbf{D}' = D_{23}$, and where $D_{23} = 0$, $\mathbf{D}' = D_{24}$; for the cubic branches $\mathbf{a}' = a_3$, $\mathbf{b}' = b_3$.)

We have now proved that every sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$, whether free or unfree, has a unique image point on the model we have constructed for $W_{2,n}$; and we are now in a position to prove conversely that every point of this model is the image of a unique sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$. This we do by induction over n, as it is trivially true for the line $W_{2,1}$. We assume it therefore for $W_{2,n-1}$. On the model of $W_{2,n}$, no two members of the pencil $|W_{2,n-1}(\mathsf{P}_1)|$ have any point in common, since whatever co-ordinates vanish there must always be at least two whose ratio is different for different values of λ. By hypothesis every point of each member of the pencil is the image of a unique sequence $\mathsf{P}_1^{(1)} \ldots \mathsf{P}_n^{(1)}$ starting from the appropriate point $\mathsf{P}_1^{(1)}$, and hence of a unique sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ containing the first neighbourhood point P_1 corresponding to this member of the pencil. The theorem is thus proved.

From now on we shall refer to the model simply as $W_{2,n}$.

9. Regular transformation

We now consider the effect on the branch (3·1) and its t-invariants of a local transformation in the plane, regular at the origin P_0, i.e. of a substitution

$$x \to X(x,y), \quad y \to Y(x,y), \tag{9·1}$$

where $X(x, y)$, $Y(x, y)$ are formal power series in (x, y), without constant terms, and whose linear forms are linearly independent, and can conveniently be taken to be the right-hand members of (3·3). Substitution from (3·1) gives

$$X(x,y) = A_1 t + A_2 t^2 + A_3 t^3 + \ldots,$$
$$Y(x,y) = B_1 t + B_2 t^2 + B_3 t^3 + \ldots,$$

where A_i, B_i are polynomials in $(a_1, b_1, \ldots, a_i, b_i)$ and the coefficients in the forms of degree $\leqslant i$ in the series $X(x, y)$, $Y(x, y)$ (these latter we shall call the transforming coefficients); and the induced transformation on any t-invariant $\mathbf{F}(a_1, b_1, \ldots, a_n, b_n)$ is of the form

$$\mathbf{F}(a_1, b_1, \ldots, a_n, b_n) \rightarrow \mathbf{F}(A_1, B_1, \ldots, A_n, B_n) \tag{9·2}$$

in which, as the regular transformation (9·1) evidently permutes with the reparametrization (3·2), the right-hand member must be a t-invariant isobaric with $\mathbf{F}$, identically in the transforming coefficients.

We now prove that the effect of the substitution (9·2) on all the principal t-invariants of rank $\leqslant n$ is to transform each of the monomials in the parametrization of $W_{2,n}$ into a linear combination of these same monomials, i.e. the regular transformation (9·1) induces a self-collineation on $W_{2,n}$. This is certainly true for $n = 1$, since $A_1 = pa_1 + qb_1$, $B_1 = ra_1 + sb_1$; we therefore assume it for any value of n and prove it for $n+1$.

We remark to begin with that the transformation (9·1) can be made in two stages, of which the second is the linear transformation (3·3), and the first is of the form

$$x \rightarrow x + X^*(x,y), \quad y \rightarrow y + Y^*(x,y), \tag{9·3}$$

where $X^*(x, y)$, $Y^*(x, y)$ are formal power series of order $\geqslant 2$, that is without constant or linear terms. This latter transformation leaves fixed every point of the first neighbourhood of P_0, and induces in the plane in which P_0 is dilated a transformation leaving fixed every point of the neighbourhood line $x^{(1)} = 0$, and regular at each such point. Thus by the inductive hypothesis, applied to the parametrization of each $W_{2,n-1}(\mathsf{P}_1)$ separately, and using the dilating substitution on this, all the co-ordinate monomials in the parametrization of $W_{2,n}$ which are (a, D) forms are replaced under (9·3) by linear combinations of themselves. But all the co-ordinate monomials are linearly expressible in terms of these by means of the linear identities, and the equations (linear in λ, which is unaltered by this transformation) of the ambient of $W_{2,n-1}(\mathsf{P}_1)$. Thus the transformation induced on all the co-ordinate monomials by (9·3) is linear, and amounts to a self-collineation of $W_{2,n}$. The second (linear) constituent of the whole transformation (9·1) is of course already known to induce a linear transformation on the co-ordinate monomials.

It is to be noted that in the case of the image point of an unfree sequence, the use of the dilating substitution in the above proof ensures (by an obvious induction over the species) that the transformation (9·1) induces on the invariants of the singular branch defining the sequence a transformation which produces just the same linear transformation on the co-ordinate monomials as in the case of the free sequence and the simple branch.

Since any simple branch at P_0 can be transformed into any other by a transformation (9·1), the group of self-collineations of $W_{2,n}$ is transitive on what we may call its ordinary points, those that are images of free sequences, and determinations of the generic point.

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We can illustrate the transformations induced by (9·1) on the t-invariants of a simple branch by giving those for the t-invariants of rank $\leqslant 4$

$$\begin{aligned}
\mathbf{a}, \mathbf{b} &\to f_1(\mathbf{a}, \mathbf{b}),\\
\mathbf{D} &\to \delta\mathbf{D} + f_3(\mathbf{a}, \mathbf{b}),\\
\mathbf{G}, \mathbf{G}_1 &\to f_1(\mathbf{G}, \mathbf{G}_1) + \mathbf{D}f_2(\mathbf{a}, \mathbf{b}) + f_5(\mathbf{a}, \mathbf{b}),\\
\mathbf{I}, \mathbf{I}_1, \mathbf{I}_2 &\to f_1(\mathbf{I}, \mathbf{I}_1, \mathbf{I}_2) + f_{1,2}(\mathbf{G}, \mathbf{G}_1; \mathbf{a}, \mathbf{b}) + \mathbf{D}f_4(\mathbf{a}, \mathbf{b}) + f_7(\mathbf{a}, \mathbf{b}),\\
\mathbf{J}, \mathbf{J}_1, \mathbf{J}_2 &\to f_1(\mathbf{J}, \mathbf{J}_1, \mathbf{J}_2) + f_{1,3}(\mathbf{I}, \mathbf{I}_1, \mathbf{I}_2; \mathbf{a}, \mathbf{b}) + f_2(\mathbf{G}, \mathbf{G}_1)\\
&\qquad + \mathbf{D}f_{1,2}(\mathbf{G}, \mathbf{G}_1; \mathbf{a}, \mathbf{b}) + \mathbf{D}f_7(\mathbf{a}, \mathbf{b}) + f_{10}(\mathbf{a}, \mathbf{b})
\end{aligned}$$

and so on; where $\delta = ps - qr$, f_i stands for a form of degree i, and $f_{i,j}$ for one of degrees i, j in two sets of arguments, with coefficients which are polynomials in the transforming coefficients.

10. General structure of $W_{2,n}$

At this point we can make a few simple remarks on the general properties of $W_{2,n}$ as a whole. In the first place $W_{2,n}$ has no singular points. We know in fact that the ruled quintic $W_{2,2}$ has none, and if $W_{2,n-1}$ has none, neither has $W_{2,n}$, being generated by the pencil $|W_{2,n-1}(\mathsf{P}_1)|$ of non-singular varieties, without base points.

Each primal of the pencil $|W_{2,n-1}(\mathsf{P}_1)|$ is of course itself generated by a pencil $|W_{2,n-2}(\mathsf{P}_2)|$; $W_{2,n}$ has thus on it an ∞^2 algebraic system $\{W_{2,n-2}(\mathsf{P}_2)\}$ of projective images of $W_{2,n-2}$, each corresponding to a particular second neighbourhood point P_2, and mapping the $(n-2)$th neighbourhood of this point, i.e. being the locus of images of sequences which have $\mathsf{P}_0\,\mathsf{P}_1\,\mathsf{P}_2$ in common. Similarly there is an algebraic ∞^3 system $\{W_{2,n-3}(\mathsf{P}_3)\}$ of projective images of $W_{2,n-3}, \ldots,$ an ∞^{n-2} system $\{W_{2,2}(\mathsf{P}_{n-2})\}$ of ruled quintics, and an ∞^{n-1} system of lines $\{W_{2,1}(\mathsf{P}_{n-1})\}$. Each of these systems generates $W_{2,n}$ simply, and is compounded with each of them that follow it in the above list.

Every point of $W_{2,n}$ is ordinary, i.e. the image of a free sequence and a determination of the generic point, except those of $n-1$ primals $\Phi_2, \ldots, \Phi_n$, where Φ_i is the locus of images of sequences of type i, in which P_i is directly proximate to P_{i-2}. Each of these primals is irreducible and non-singular; and each set of r of them intersect properly in an irreducible, non-singular, and non-vacuous V_{n-r}, locus of images of sequences in which the corresponding points are all indirectly proximate to their penultimate predecessors. Thus on $W_{2,3}$ the ruled septimic surface Φ_2 parametrized in (8·3) and the ruled surface Φ_3 of order 15 parametrized in (8·6) intersect in the sextic curve $\Phi_{2,3}$, parametrized in (8·7), locus of images of sequences in which P_2 is indirectly proximate to P_0 and also P_3 to P_1. We shall denote the intersection of $\Phi_{i_1}, \ldots, \Phi_{i_r}$, locus of images of sequences of type $i_1 \ldots i_r$, by $\Phi_{i_1 \ldots i_r}$.

There is also on Φ_i a $V_{r-2}\Phi_{(i,i+1)}$, locus of images of sequences of type $(i, i+1)$ in which P_i, P_{i+1} are both indirectly proximate to P_{i-2}; on this again a $V_{r-3}\Phi_{(i,i+1,i+2)}$, and so on, $\Phi_{(i,\ldots,i+s)}$ being the locus of images of sequences of type $(i, \ldots, i+s)$, in which $\mathsf{P}_i, \ldots, \mathsf{P}_{i+s}$ are all indirectly proximate to P_{i-2}. $\Phi_{(i,\ldots,i+s)}$ lies in Φ_i and does not meet $\Phi_{i+1}, \ldots, \Phi_{i+s}$, but meets each of the remaining Φ_j's in a variety of the appropriate dimensions.

It can be observed in the cases $n = 2, 3$ with which we have dealt in detail, and can be proved in general by an easy induction from the way in which the parametrization is obtained recursively, from one value of n to the next, that the equations of Φ_i consist in the vanishing of all the co-ordinates except those which, with each monomial factor of rank $> i$, contain the t-invariants of rank i in the highest degree. Thus Φ_n is given by the vanishing of all co-ordinates which do not contain the invariants of rank n, Φ_{n-1} by the vanishing of all except those which either contain the invariants of rank n and are linear in those of rank $n-1$, or do not contain those of rank n and are quartic in those of rank $n-1$, and so on.

Each of $\Phi_2, \ldots, \Phi_n$ is invariant under the whole group of self-collineations of $W_{2,n}$.

We can define also a primal $\Phi_{\overline{2}}$, on this a $V_{n-2}\Phi_{\overline{23}}$, on this again a $V_{n-3}\Phi_{\overline{234}}$, and so on, where $\Phi_{\overline{2\ldots i}}$ is the locus of images of sequences in which $\mathsf{P}_2, \ldots, \mathsf{P}_i$ are collinear with $\mathsf{P}_0\mathsf{P}_1$. The conditions for this collinearity are the vanishing of all the t-invariants of rank $2, \ldots, i$, and the locus is obtained by putting these zero values into the generic point; in particular $\Phi_{\overline{2\ldots n}}$ is the curve of order $m_n = \frac{1}{2}(3^n-1)$, parametrized by the monomials of this degree in $(\mathbf{a}, \mathbf{b})$ only among the co-ordinates of the generic point, with all the other co-ordinates equal to zero. These loci are not of course invariant under all the self-collineations of $W_{2,n}$: $\Phi_{\overline{2\ldots i}}$ is in fact transformed by the collineation induced by (9·1) into an equivalent variety, locus of images of sequences in which the points $\mathsf{P}_0, \ldots, \mathsf{P}_i$ are on some curve of a fixed pencil with a simple base point at P_0, image under (9·1) of the pencil of lines through P_0.

There is on $W_{2,n}$ a unique line l_1, unisecant to the pencil $|W_{2,n-1}(\mathsf{P}_1)|$, given by the vanishing of all the co-ordinates except the two equated to $\mathbf{a}\mathbf{D}^{m_{n-1}}$, $\mathbf{b}\mathbf{D}^{m_{n-1}}$. l_1 is in fact the locus $\Phi_{(2\ldots n)}$, of images of sequences in which $\mathsf{P}_1, \ldots, \mathsf{P}_n$ are all proximate to P_0; for its intersection with each $W_{2,n-1}(\mathsf{P}_1)$ is given, in the parametrization of the latter, by the vanishing of all the co-ordinate monomials except $d^{m_{n-1}}$, and this represents the sequence $\mathsf{P}_1^{(1)} \ldots \mathsf{P}_n^{(1)}$ collinear along the neighbourhood line $x^{(1)} = 0$.

It follows inductively from this that there are on $W_{2,n}$ ∞^1 lines $\{l_2\}$, one on each member of the pencil $|W_{2,n-1}(\mathsf{P}_1)|$, unisecant to the pencil $|W_{2,n-2}(\mathsf{P}_2)|$ on $W_{2,n-1}(\mathsf{P}_1)$, and generating the surface $\Phi_{(3\ldots n)}$; ∞^2 lines $\{l_3\}$, one on each $W_{2,n-2}(\mathsf{P}_2)$, generating the threefold $\Phi_{(4\ldots n)}$; $\ldots$; ∞^{n-2} lines $\{l_{n-1}\}$, which are the directrices of the ruled quintics $W_{2,2}(\mathsf{P}_{n-2})$ and generate Φ_n; and ∞^{n-1} lines $\{l_n\} = \{W_{2,1}(\mathsf{P}_{n-1})\}$, generating $W_{2,2}$ itself.

The pencil $|W_{2,n-1}(\mathsf{P}_1)|$ and the $n-1$ primals $\Phi_2, \ldots, \Phi_n$ form a base for primals on $W_{2,n}$. This we prove by induction, as it is true for the ruled quintic $W_{2,2}$, on which the generators $|W_{2,1}(\mathsf{P}_1)|$ and the directrix line Φ_2 are a base for curves. We therefore assume the result for $W_{2,n-1}$. On $W_{2,n}$ the intersections $(\Phi_2 . W_{2,n-1}(\mathsf{P}_1)), \ldots, (\Phi_n . W_{2,n-1}(\mathsf{P}_1))$ are a particular $W_{2,n-2}(\mathsf{P}_2)$ and the images of $\Phi_2, \ldots, \Phi_{n-1}$ on $W_{2,n-1}$. Thus if Ω is any primal on $W_{2,n}$, the intersection $(\Omega . W_{2,n-1}(\mathsf{P}_1))$ is by hypothesis equivalent on $W_{2,n-1}(\mathsf{P}_1)$ to some linear combination $\sum_{i=2}^{n} \alpha_i(\Phi_i . W_{2,n-1}(\mathsf{P}_1))$; and as $|W_{2,n-1}(\mathsf{P}_1)|$ is a pencil without base points or reducible members, this means that Ω itself is equivalent on $W_{2,n}$ to $\alpha_1 W_{2,n-1}(\mathsf{P}_1) + \sum_{i=2} \alpha_i \Phi_i$, which is the theorem.

A similar argument shows that the intersections r at a time of $\Phi_2, \ldots, \Phi_n$ and $W_{2,n-1}(\mathsf{P}_1)$ are a base for V_{n-r}'s on $W_{2,n}$. For $r = n-1$, however, a neater result is that the lines $l_1, \ldots, l_n$ are a base for curves. This again is true for the ruled quintic $W_{2,2}$, the directrix being l_1 and the generators $|l_2|$; and assuming it to be true for $W_{2,n-1}$, if any curve C on $W_{2,n}$ meets the generic $W_{2,n-1}(\mathsf{P}_1)$ in β_1 points, $C - \beta_1 l_1$ is equivalent to some curve on $W_{2,n-1}(\mathsf{P}_1)$, i.e. by hypothesis to $\sum_{i=2}^{n} \beta_i l_i$.

To make ideas concrete, we illustrate below the various special loci on $W_{2,3}$, and the intersections of those on $W_{2,4}$ with a general $W_{2,3}(\mathsf{P}_1)$. (Figures 1 and 2.)

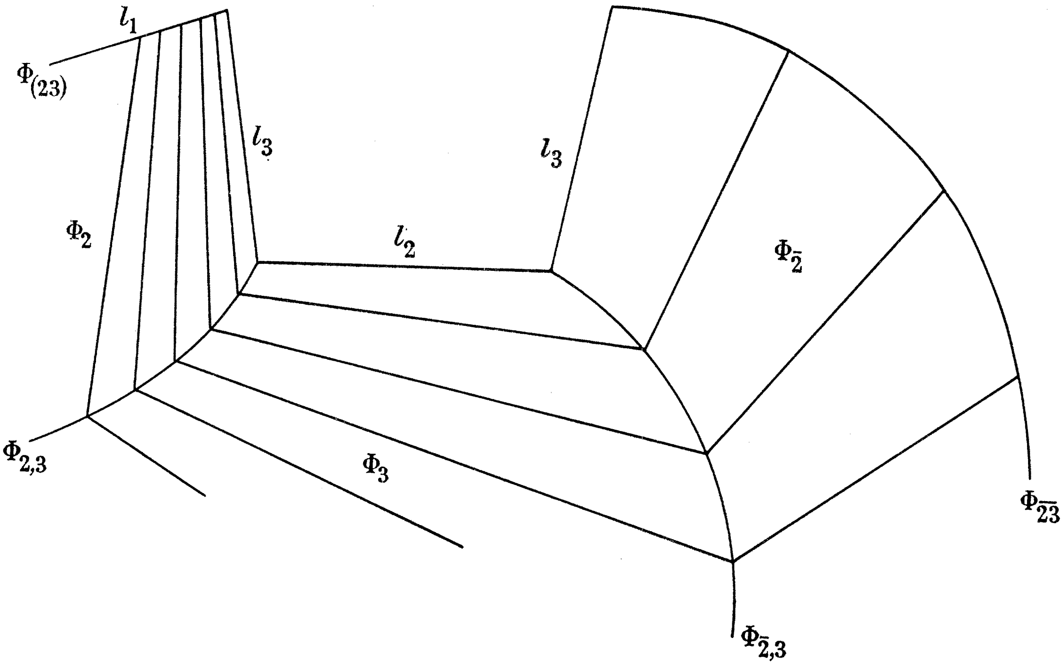


FIGURE 1. Special loci on $W_{2,3}$.

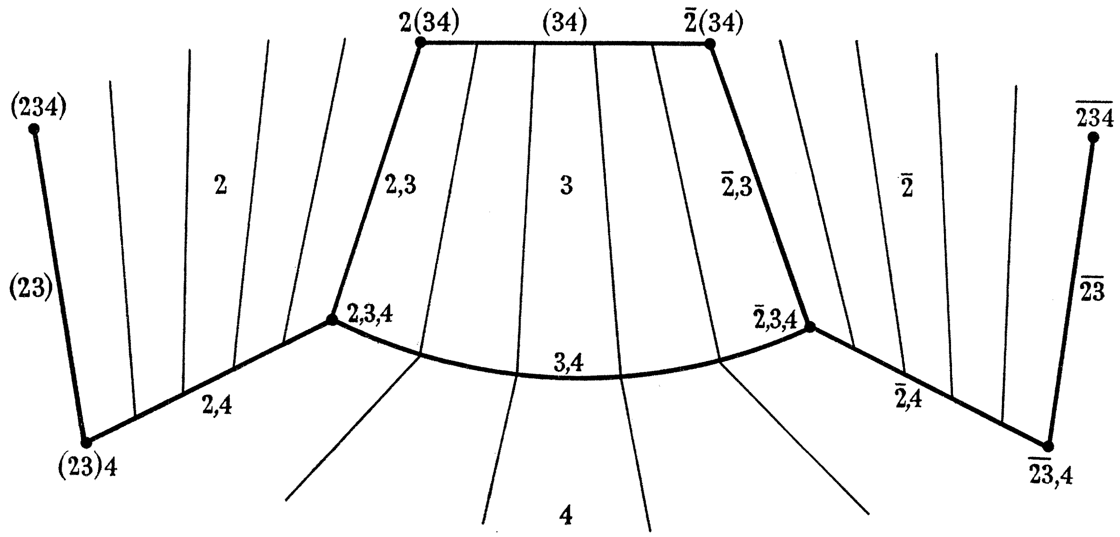


FIGURE 2. Special loci on $W_{2,4}$ (traces on $W_{2,3}(\mathsf{P}_1)$).

11. THE SPECIAL LOCI ON $W_{2,n}$

It is possible to form some idea of the geometrical relations of the special or Φ loci on $W_{2,n}$, at least of those that are not of too high dimensions, for any value of n, without actually completing the parametrization of $W_{2,n}$ as a whole.

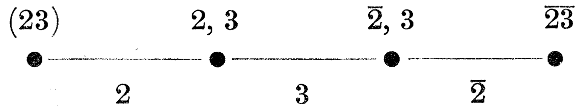
Figure 1 shows the Φ loci on $W_{2,3}$. $\Phi_{(23)}$, $\Phi_{2,3}$, $\Phi_{\bar{2},3}$, and $\Phi_{\overline{23}}$ are rational curves, unisecant to $W_{2,2}(\mathsf{P}_1)$; and it is clear that this will be the case whenever the proximity symbol used as suffix to the Φ contains all the indices $2, \ldots, n$, since in such a sequence all the points except

P_0, P_1 are completely determined by these two, being either collinear with them or satellites. $\Phi_{(23)} = l_1$ is parametrized in (8·4), $\Phi_{2,3}$ is the sextic curve (8·7), $\Phi_{\bar{2},3}$ is obtained by putting $\mathbf{D}'' = 0$ in (8·6) and is of order 9, and $\Phi_{\overline{23}}$ is obtained by putting $\mathbf{D} = \mathbf{G} = \mathbf{G}_1 = 0$ in (7·5), so that its equations are

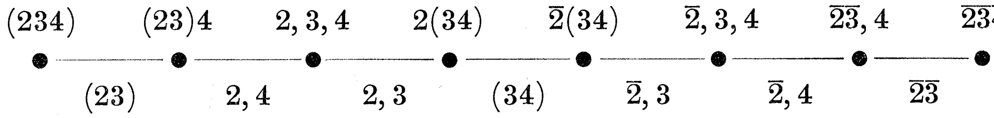
$$Y_{i4} = \mathbf{a}^{13-i}\mathbf{b}^i \quad (i = 0, \ldots, 13)$$

and the vanishing of all the other co-ordinates. The ruled surfaces Φ_2, Φ_3, $\Phi_{\bar{2}}$ each have two consecutive of these four curves as directrices, so that their orders are $1+6 = 7$, $6+9 = 15$, and $9+13 = 22$. In the figure, each generator l_2 of Φ_3, and the generators l_3 of Φ_2, $\Phi_{\bar{2}}$ that meet it, are the directrix line and two generators of a ruled quintic of the pencil $W_{2,2}(\mathsf{P}_1)$.

Figure 2 shows the intersections of the Φ loci on $W_{2,4}$ with a typical $W_{2,3}(\mathsf{P}_1)$. We see that the traces of $\Phi_{(34)}$, Φ_3, $\Phi_{3,4}$, Φ_4 are the images of $\Phi_{(23)}$, Φ_2, $\Phi_{2,3}$, Φ_3 on $W_{2,3}$, since of course the second and third implicit points in the sequence $\mathsf{P}_1^{(1)} \ldots \mathsf{P}_1^{(1)}$ are the images of the third and fourth in the sequence $\mathsf{P}_0 \ldots \mathsf{P}_4$. On the other hand the traces of Φ_2, $\Phi_{\bar{2}}$ are two particular members of the pencil $W_{2,2}(\mathsf{P}_2)$ on $W_{2,3}(\mathsf{P}_1)$, and the traces of the other Φ loci, having either 2 or $\bar{2}$ amongst their suffixes, are just the copies on these two surfaces of the traces on $W_{2,2}(\mathsf{P}_1)$ of the Φ loci on $W_{2,3}$. Thus the two point and line chains, copies of the chain



on the traces of Φ_2, $\Phi_{\bar{2}}$, are joined by the line l_2, trace of $\Phi_{(34)}$, to form a single chain



on $W_{2,3}(\mathsf{P}_1)$. This means that on $W_{2,4}$ we have a sequence of ruled surfaces $\Phi_{(23)}$, $\Phi_{2,4}$, $\Phi_{2,3}$, $\Phi_{(34)}$, $\Phi_{\bar{2},3}$, $\Phi_{\bar{2},4}$, $\Phi_{\overline{23}}$, each of which has a generator in each member of the pencil $W_{2,3}(\mathsf{P}_1)$, and as directrix curves two consecutive members of the sequence $\Phi_{(234)}$, $\Phi_{(23)4}$, $\Phi_{2,3,4}$, $\Phi_{2(34)}$, $\Phi_{\bar{2}(34)}$, $\Phi_{\bar{2},3,4}$, $\Phi_{\overline{23},4}$, $\Phi_{\overline{234}}$. The orders of these increase along the sequence from 1 to $m_4 = \frac{1}{2}(3^4-1) = 40$.

These orders can of course be found from the parametrization (7·8), when the curves themselves have been identified in terms of the co-ordinate system there used; but it is simpler to note that each of these seven ruled surfaces, since it only contains one independent absolute t-invariant besides that determining P_1, must have a parametrization in terms of a_m, b_m, $D_{mm'}$ only, where a_m, b_m are the first pair a_i, b_i that do not both vanish, and $D_{mm'}$ the first D_{mj} that does not vanish, for the corresponding singular branch. Considerations of weight show that the parametrization of the ruled surface must be of the form

$$a_m^g D_{mm'}^h : \ldots : b_m^g D_{mm'}^h : a_m^{g+k} : \ldots : b_m^{g+k},$$

where $k/h = (m+m')/m$. Thus the difference in order between the two directrix curves is k. Further, it can be seen that h, k are mutually prime, so that $k = (m+m')/(m, m')$, where (m, m') denotes the highest common factor of m, m'. For if m, m' have a common factor, the ruled surface in question is a birational image of that on $W_{2,n'}$ (for a suitable $n' < n$) representing the shorter and simpler sequences obtained by omitting all points after the one free point (since these sequences are in one-one correspondence with those of the type in question); and for the typical branch through this shorter sequence, m, m' are proportional to their values here, but are mutually prime.

In table 1 each entry in the first column gives the proximity type symbol corresponding to one of the ruled Φ surfaces on $W_{2,4}$; in the second column are the multiplicities of $\mathsf{P}_0 \ldots \mathsf{P}_4$ on a typical branch through the sequence; in the third, the values of m (the multiplicity of P_0) and m' (the sum of the multiplicities of points of the sequence on the line $\mathsf{P}_0\mathsf{P}_1$; in the fourth that of $k = (m+m')/(m, m')$; in the fifth and sixth, the type symbol of the sequence corresponding to a Φ curve on $W_{2,4}$, and the order of this curve; and in the last column the order of the ruled Φ surface. The entries in the last two columns are obtained from the fact that each entry in the fourth column is the difference, and each in the last the sum, of consecutive entries in the sixth column (and of course that l_1 is a line, which fixes the first entry in the sixth column). As a check on the method we note that $\Phi_{\overline{234}}$ is a curve of order $m_4 = 40$, as we expect.

TABLE 1

				(234)	1	
(23)	3, 1, 1, 1, 1	3, 4	7			9
				(23)4	8	
2, 4	4, 2, 2, 1, 1	4, 6	5			21
				2, 3, 4	13	
2, 3	3, 2, 1, 1, 1	3, 5	8			34
				2(34)	21	
(34)	3, 3, 1, 1, 1	3, 6	3			45
				$\bar{2}$(34)	24	
$\bar{2}$, 3	2, 2, 1, 1, 1	2, 5	7			55
				$\bar{2}$, 3, 4	31	
$\bar{2}$, 4	2, 2, 2, 1, 1	2, 6	4			66
				$\overline{23}$, 4	35	
$\overline{23}$	1, 1, 1, 1, 1	1, 4	5			75
				$\overline{234}$	40	

The only Φ surface on $W_{2,4}$ that is not ruled is $\Phi_{3,4}$; this is a birational image of $W_{2,2}$, on which the directrix line appears as the 13-ic curve $\Phi_{2,3,4}$, and the generators as the sextic curves traced by $|W_{2,3}(\mathsf{P}_1)|$.

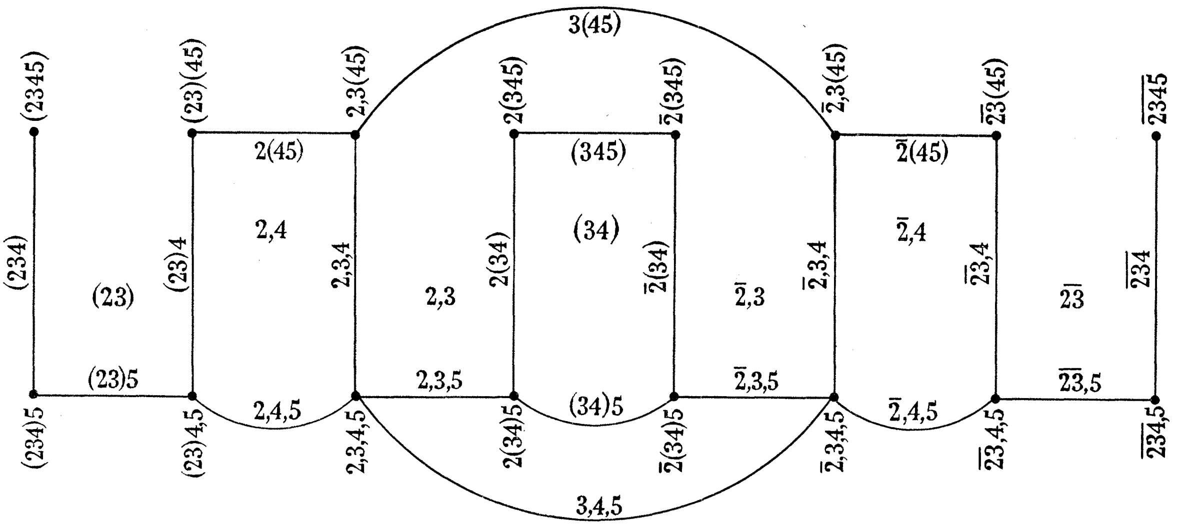


FIGURE 3. Special loci on $W_{2,5}$ (traces on $W_{2,4}(\mathsf{P}_1)$).

This method of deriving the chain of Φ curves and ruled Φ surfaces on $W_{2,4}$ from that on $W_{2,3}$, by taking two copies of the chain of traces on $W_{2,2}(\mathsf{P}_1)$ and linking their ends together by the line l_2 to give the chain on $W_{2,3}(\mathsf{P}_1)$, can equally be used to obtain the chain on $W_{2,n}$ from that on $W_{2,n-1}$. Figure 3 is obtained in this way from a simplified version of figure 2, and figure 4 from figure 3.

Two copies of the point and line chain for $W_{2,n-1}$ are placed side by side, each of the indices in all the proximity symbols being increased by unity, with 2 prefixed to all the symbols in one copy and $\bar{2}$ to all those in the other; where 2 is prefixed to $\bar{3}$, $\overline{34}$, ..., it becomes (23), (234), ... The end-points $2(3\ldots n)$, $\bar{2}(3\ldots n)$ of the two chains are then linked by a new line l_2, trace of the ruled surface $\Phi_{(3\ldots n)}$. There are thus 2^{n-1} Φ curves, and $2^{n-1}-1$ ruled Φ surfaces, with a generator in each $W_{2,n-1}(\mathsf{P}_1)$. The orders of these can be found by the method used above for $W_{2,4}$; in the case of $W_{2,5}$ they are

1, 10, 17, 29, 34, 47, 55, 66, 69, 79, 86, 97, 101, 110, 115, 121

11, 27, 46, 63, 81, 102, 121, 135, 148, 165, 183, 198, 211, 225, 236

(the curves in the upper row and ruled surfaces in the lower, proceeding along the chain from $\Phi_{(2345)}$ to $\Phi_{\overline{2345}}$).

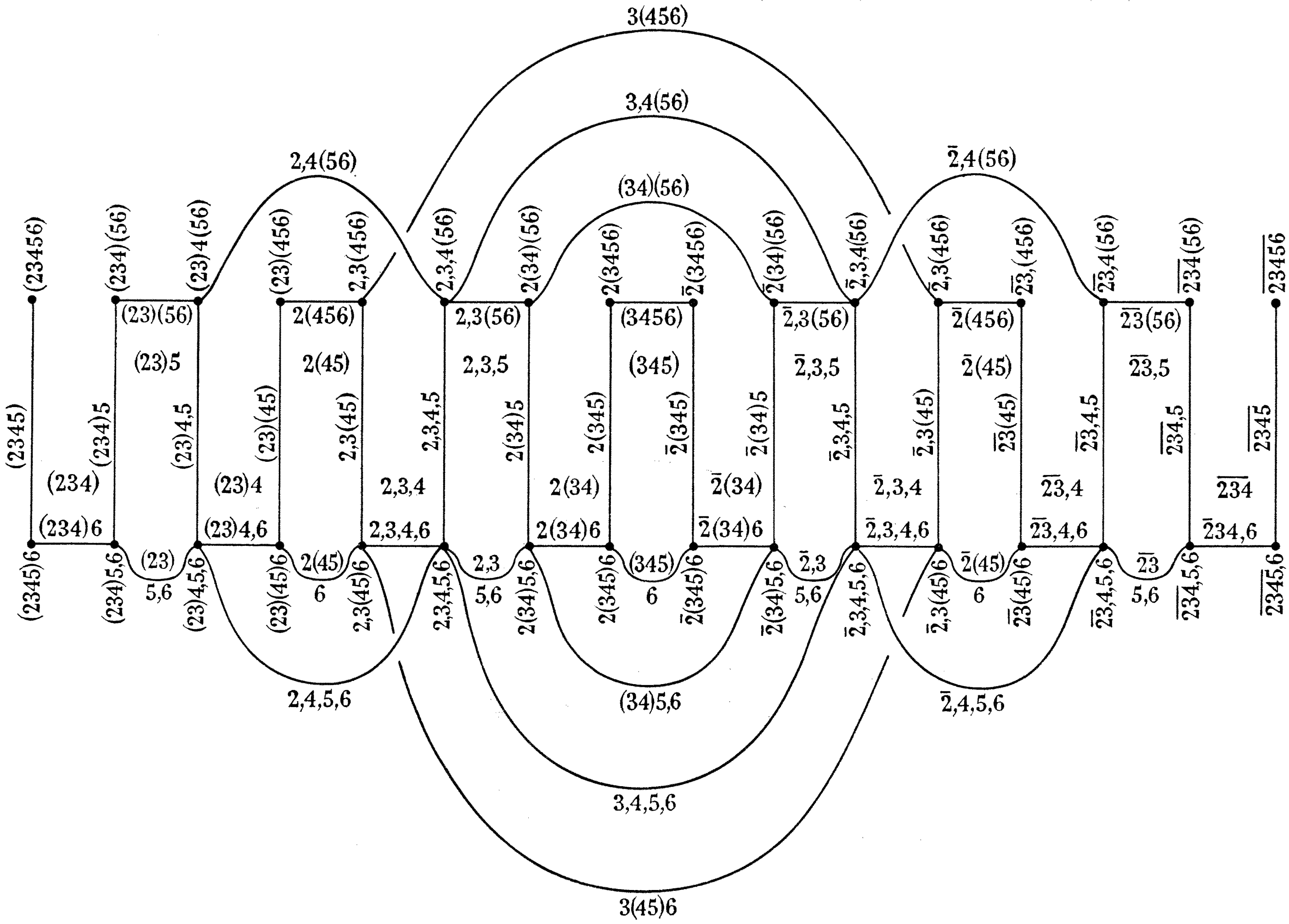


FIGURE 4. Special loci on $W_{2,6}$ (traces on $W_{2,5}$ (P_1)).

There are also non-ruled Φ surfaces tracing curves instead of lines on $W_{2,n-1}(\mathsf{P}_1)$. Besides the two copies of each such curve in the previous figure, there are also the images of those of the Φ curves of $W_{2,n-1}$ which have 2, not $\bar{2}$, in their symbol; these have all the indices increased by unity without 2 or $\bar{2}$ being added. Thus in figure 3, the curve traces of $\Phi_{(34)5}$, $\Phi_{3,4,5}$, $\Phi_{3(45)}$, are the images of the curves $\Phi_{(23)4}$, $\Phi_{2,3,4}$, $\Phi_{2(34)}$ on $W_{2,4}$.

We can also identify in the diagram a number of surfaces which are the traces of threefold Φ loci. We note that every alternate (odd numbered) line in the chain is an l_n, every alternate one of the remainder is an l_{n-1}, and so on, the middle line of all being an l_2. Every two consecutive l_n's are two generators, and the line between them in the chain is directrix line, of such a ruled surface trace of a threefold Φ locus (if the diagrams are drawn in the 'battlement' pattern used here, these are the open rectangles of the pattern, and their symbols are written inside them in figures 3 and 4). The ruled surfaces whose directrix lines are l_{n-1} are ruled quintics of the system $W_{2,2}(\mathsf{P}_{n-2})$; of the rest, those whose directrix lines are l_{n-i} are of order $2i+3$, being images of the ruled surface $\Phi_{(2\ldots i)}$ on $W_{2,i+1}$. There are also the images of those Φ surfaces on $W_{2,n-1}$ which have 2, not $\bar{2}$, in their symbols, and of course the copies of surfaces similarly arising in the previous diagrams. In figure 3, we detect the following surfaces, ruled except for the fourth, which is a birational image of $W_{2,2}$; in table 2 the second column gives the symbol of the minimum directrix curve, the third (where applicable) that of a second directrix curve, and the last two of two generators (or images of generators).

TABLE 2

$\Phi_{3,5}$	(34) 5	3, 4, 5	2, 3, 5	$\bar{2}$, 3, 5
$\Phi_{3,4}$	3, 4, 5	3(45)	2, 3, 4	$\bar{2}$, 3, 4
$\Phi_{(45)}$	3(45)	—	2(45)	$\bar{2}$(45)
$\Phi_{4,5}$	3, 4, 5	—	2, 4, 5	$\bar{2}$, 4, 5
$\Phi_{2,5}$	2, 4, 5	—	2, 3, 5	(23) 5
$\Phi_{\bar{2},5}$	$\bar{2}$, 4, 5	—	$\bar{2}$, 3, 5	$\overline{23}$, 5

A similar study of the Φ loci on $W_{2,6}$, assisted by figure 4, is left as an exercise for the reader.

Traces of Φ loci of higher dimensions can also be picked out in the figures, but of course much less clearly.

12. BASE AND INTERSECTION THEORY ON $W_{2,4}$

In a paper dealing in detail with $W_{2,3}$ (Du Val 1961) I have shown that on this (denoting the prime sections by $|\Pi|$, and abbreviating $W_{2,2}(\mathsf{P}_1)$ to W)

$$\Pi = 13W + 4\Phi_2 + \Phi_3, \quad \Pi.\Pi = 5l_1 + 22l_2 + 81l_3,$$

so that the order of $W_{2,3}$ is 108; the intersection table is

	W	Φ_2	Φ_3
W	0	l_3	l_2
Φ_2	l_3	$-3l_3$	l_1+5l_3
Φ_3	l_2	l_1+5l_3	$-3l_1-4l_2-15l_3$
l_1	1	-3	0
l_2	0	1	-3
l_3	0	0	1

and the remaining Φ loci satisfy the equivalences

$$\Phi_{\bar{2}} = 3W + \Phi_2, \quad \Phi_{(23)} = l_1, \quad \Phi_{2,3} = l_1 + 5l_3,$$

$$\Phi_{\bar{2},3} = l_1 + 3l_2 + 5l_3, \quad \Phi_{\overline{23}} = l_1 + 3l_2 + 9l_3.$$

It is perhaps worth setting down here briefly the corresponding theory for $W_{2,4}$, as this illustrates the way in which the properties of $W_{2,n}$ can be obtained from those of $W_{2,n-1}$.

A base for primals on $W_{2,4}$ consists of $\mathfrak{w} = W_{2,3}(\mathsf{P}_1)$, Φ_2, Φ_3, Φ_4, and one for curves of l_1, l_2, l_3, l_4; one for surfaces would consist of the intersections of these by pairs, but we shall replace $\Phi_{2,3}$, $\Phi_{3,4}$ by $\Phi_{(23)}$, $\Phi_{(34)}$, which are of lower order, and enable us to express all the surfaces in which we are interested without negative coefficients. We shall denote the traces of Φ_4, Φ_3, Φ_2 on $\mathfrak{w}$ by U, V, $W = W_{2,2}(\mathsf{P}_2)$.

As the line system $\{l_4\}$ is mapped one-one on $W_{2,3}$, every subvariety of $W_{2,3}$ has an image in $\{l_4\}$, the variety (of one more dimension) generated by the corresponding subsystem of $\{l_4\}$. It has also an image on Φ_4, which is a birational image of $W_{2,3}$, being unisecant to $\{l_4\}$, and of course an image on each $\mathfrak{w}$. These images are tabulated below:

$W_{2,3}$:	W	Φ_2	Φ_3	l_1	l_2	l_3	$\Phi_{2,3}$
$\{l_4\}$:	$\mathfrak{w}$	Φ_2	Φ_3	$\Phi_{(23)}$	V	W	$\Phi_{2,3}$
Φ_4:	U	$\Phi_{2,4}$	$\Phi_{3,4}$	$\Phi_{(23)4}$	k	l_3	$\Phi_{2,3,4}$
$\mathfrak{w}$:	W	V	U	l_2	l_3	l_4	k

where k denotes the sextic curve seen in figure 2 as the trace of $\Phi_{3,4}$ on $\mathfrak{w}$. The images of $\Phi_{\bar{2}}$, $\Phi_{\bar{2},3}$, $\Phi_{\overline{23}}$ in $\{l_4\}$ are the similarly named condition loci on $W_{2,4}$, and their images on Φ_4 are the same with an additional suffix 4 in each case; those on $\mathfrak{w}$ are without geometrical significance for $W_{2,4}$. Equivalences on $W_{2,3}$ hold also between the images in all three systems, and the image in $\{l_4\}$ of an intersection on $W_{2,3}$ is the corresponding intersection on $W_{2,4}$.

Since the difference between two directrix curves of a ruled surface is a multiple of the generator, we obtain at once the following equivalences for curves

$$\Phi_{(234)} = l_1, \quad \Phi_{(23)4} = l_1 + 7l_4, \quad \Phi_{2,3,4} = l_1 + 5l_3 + 7l_4,$$
$$\Phi_{2(34)} = l_1 + 5l_3 + 15l_4, \quad \Phi_{\bar{2}(34)} = l_1 + 3l_2 + 5l_3 + 15l_4,$$
$$\Phi_{\bar{2},3,4} = l_1 + 3l_2 + 5l_3 + 22l_4, \quad \Phi_{\overline{23},4} = l_1 + 3l_2 + 9l_3 + 22l_4,$$
$$\Phi_{\overline{234}} = l_1 + 3l_2 + 9l_3 + 27l_4, \quad k = l_2 + 5l_4.$$

Similarly the prime sections of U, V, W, and the Φ surfaces are immediately found to be

$$U\colon\ l_2 + 9l_3 + 5l_4, \quad V\colon\ l_2 + 6l_4, \quad W\colon\ l_3 + 4l_4,$$
$$\Phi_{(23)}\colon\ l_1 + 8l_4, \quad \Phi_{2,4}\colon\ l_1 + 13l_3 + 7l_4, \quad \Phi_{2,3}\colon\ l_1 + 5l_3 + 28l_4,$$
$$\Phi_{(34)}\colon\ l_1 + 24l_2 + 5l_3 + 15l_4, \quad \Phi_{\bar{2},3}\colon\ l_1 + 3l_2 + 5l_3 + 46l_4,$$
$$\Phi_{\bar{2},4}\colon\ l_1 + 3l_2 + 40l_3 + 22l_4, \quad \Phi_{\overline{23}}\colon\ l_1 + 3l_2 + 9l_3 + 62l_4,$$
$$\Phi_{3,4}\colon\ 6\Phi_{2,3,4} + 31k = 6l_1 + 31l_2 + 30l_3 + 197l_4,$$

the last being obtained from the fact that $\Phi_{3,4}$ is a birational image of the ruled quintic $W_{2,2}$ with $\Phi_{2,3,4}$, k representing the directrix and generator; its order is accordingly 264. From equivalences in $\{l_4\}$ we obtain

$$\Phi_{2,3} = 5W + \Phi_{(23)}, \quad \Phi_{\bar{2},3} = 3V + 5W + \Phi_{(23)}, \quad \Phi_{\overline{23}} = 3V + 9W + \Phi_{(23)},$$

and on Φ_4
$$\Phi_{\bar{2},4} = 3U + \Phi_{2,4}.$$

Now as Φ_2 is generated by lines l_4 meeting $\Phi_{2,4}$, the residual section by a prime through $\Phi_{2,4}$ is ruled in $\{l_4\}$, and from equivalence and intersection relations in this subsystem of $\{l_4\}$, as mapped on $\Phi_{2,4}$, we find (denoting the prime sections of $W_{2,4}$ by $|\Pi|$) that

$$\Pi . \Phi_2 = 21W + 4\Phi_{(23)} + \Phi_{2,4}.$$

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TABLE 3

	$\mathfrak{w}$ section	Φ_2 section	Φ_3 section	Φ_4 section	prime section	order
$\mathfrak{w}$	0	W	V	U	$13W+4V+U$	108
Φ_2	W	$-3W$	$5W+\Phi_{(23)}$	$\Phi_{2,4}$	$21W+4\Phi_{(23)}+\Phi_{2,4}$	162
Φ_3	V	$5W+\Phi_{(23)}$	$-15W-4V-3\Phi_{(23)}$	$25W+7V+5\Phi_{(23)}+\Phi_{(34)}$	$30W+31V+6\Phi_{(23)}+\Phi_{(34)}$	466
Φ_4	U	$\Phi_{2,4}$	$25W+7V+5\Phi_{(23)}+\Phi_{(34)}$	$-75W-21V-4U-15\Phi_{(23)}-4\Phi_{2,4}-3\Phi_{(34)}$	$25W+7V+36U+5\Phi_{(23)}+9\Phi_{2,4}+\Phi_{(34)}$	993
U	0	l_3	l_2+5l_4	$-3l_2-4l_3-15l_4$	$l_2+9l_3+5l_4$	15
V	0	l_4	$-3l_4$	l_2+5l_4	l_2+6l_4	7
W	0	0	l_4	l_3	l_3+4l_4	5
$\Phi_{(23)}$	l_4	$-3l_4$	0	l_1+7l_4	l_1+8l_4	9
$\Phi_{2,4}$	l_3	$-3l_3$	$l_1+5l_3+7l_4$	$-3l_1-8l_3-21l_4$	$l_1+13l_3+7l_4$	21
$\Phi_{(34)}$	l_2	$l_1+5l_3+15l_4$	$-3l_1+4l_2-15l_3-45l_4$	0	$l_1+24l_2+5l_3+15l_4$	45
l_1	1	-3	0	0	1	1
2	0	1	-3	0	1	1
3	0	0	1	-3	1	1
4	0	0	0	1	1	1

	U	V	W	$\Phi_{(23)}$	$\Phi_{2,4}$	$\Phi_{(34)}$
U	0	0	0	1	-3	0
W	0	0	0	0	1	-3
W	0	0	0	0	0	1
$\Phi_{(23)}$	1	0	0	0	-3	0
$\Phi_{2,4}$	-3	1	0	-3	-9	0
$\Phi_{(34)}$	0	-3	1	0	0	169

$$\Pi = 40\mathfrak{w}+13\Phi_2+4\Phi_3+\Phi_4,$$
$$\Pi.\Pi = 76U+291V+938W+81\Phi_{(23)}+22\Phi_{2,4}+5\Phi_{(34)},$$
$$\Pi.\Pi.\Pi = 108l_1+487l_2+1933l_3+6755l_4,$$

order of $W_{2,4}$: 9283.

Similarly, as Φ_3 is generated by lines l_4 meeting $\Phi_{(34)}$ and $\Phi_{3,4}$,

$$\Pi . \Phi_3 = 31V + 6(5W + \Phi_{(23)}) + \Phi_{(34)}$$
$$= 24V + 5W + \Phi_{(23)} + \Phi_{3,4}$$

whence incidentally $$\Phi_{3,4} = 7W + 5(5W + \Phi_{(23)}) + \Phi_{(34)}.$$

And from the equivalence and intersection relations on Φ_4 as image of $W_{2,3}$, since the images of l_1, l_2, l_3 are of orders 9, 6, 1,

$$\Pi . \Phi_4 = 36U + 9\Phi_{2,4} + \Phi_{3,4}.$$

Finally, as the residual section of $W_{2,4}$ by a prime through Φ_4 is compounded with $\{l_4\}$ and has 1, 1, and 4 generators in common with $\Phi_{(23)}$, V, and W, respectively, we see that

$$\Pi = 40\mathfrak{w} + 13\Phi_2 + 4\Phi_3 + \Phi_4. \tag{12·1}$$

The intersection table (Table 3) is now easily constructed. Most of the intersections are obvious, except those of a Φ locus with something lying on it, and these are found from the linear identity between the five columns of the main table which expresses the relation (12·1). The orders of the various loci, entered in the last column, are obtained recursively by adding, for each, the orders of the components of its prime section.

It is obvious that we could go on, using these results, to study the geometry of $W_{2,5}$, $W_{2,6}$, ... in turn, if it were worth the trouble, which obviously increases very rapidly with n. A few results clearly generalize: it is not hard to prove inductively for instance that the prime sections of $W_{2,n}$ are equivalent to

$$\tfrac{1}{2}(3^n - 1)\, W_{2,n-1}(\mathsf{P}_1) + \sum_{i=2}^{n} \tfrac{1}{2}(3^{n-i+1} - 1)\, \Phi_i,$$

that $$\Phi_{\overline{2\ldots n}} = \sum_{i=1}^{n} 3^{i-1} l_i,$$

and that the intersection matrix of primals with lines in the base has 1 everywhere in the diagonal, bordered by -3 above, and 0 everywhere else. But much in the way of further information looks as though it would be very laborious, though quite straightforward, to find.

PART II. $W_{r,n}(r \geq 3)$; IN PARTICULAR $W_{3,3}$

13. INVARIANTS OF A BRANCH IN SPACE

If we now consider a branch in S_3, with generic point

$$\left.\begin{aligned} x &= a_1 t + a_2 t^2 + a_3 t^3 + \ldots, \\ y &= b_1 t + b_2 t^2 + b_3 t^3 + \ldots, \\ z &= c_1 t + c_2 t^2 + c_3 t^3 + \ldots, \end{aligned}\right\} \tag{13·1}$$

where of course (x, y, z) are an affine co-ordinate system in S_3 with origin at the origin P_0 of the branch, and seek for its t-invariants, we have first of all those already found for the plane branch (3·1), with the adjunction of further tensor companions, since the affine transformation (3·3) is now to be replaced by a general linear transformation on (x, y, z). Thus $D_{ij} = a_i b_j - a_j b_i$ may be denoted by $D_{z(ij)}$, and we similarly define

$$D_{x(ij)} = b_i c_j - b_j c_i, \quad D_{y(ij)} = c_1 a_j - c_j a_i;$$

so that
$$\mathbf{D}_x = D_{x(12)}, \quad \mathbf{D}_y = D_{y(12)}, \quad \mathbf{D}_z = D_{z(12)}$$

are t-invariants of rank 2, and the components of a covariant vector.

As we have already so many numerical suffixes to take into account, we shall use x, y, z as the co-ordinate indices of tensors, rather than 0, 1, 2 or 1, 2, 3; indeterminate tensor indices will be denoted by Greek letters α, β, ...; and in order to apply this notation to the coefficients in (13·1) we shall write p^x, p^y, p^z for a, b, c where convenient, with the numerical suffixes unaltered (and still denoted by i, j, ... when indeterminate.)

With this notation we see that all the t-invariant (a, D) forms of part I become components of tensors, in which all the covariant indices are z and all the contravariant indices x; what we have hitherto called the tensor companions of these are merely those other components in which some or all of the contravariant indices are y instead of x. All components will now be t-invariants. Thus $\mathbf{G}$, $\mathbf{G}_1$ are the components $\mathbf{G}^x_z$, $\mathbf{G}^y_z$ of a t-invariant tensor
$$\mathbf{G}^\alpha_\beta = -2p^\alpha_2 D_{\beta(12)} + p^\alpha_1 D_{\beta(13)}$$
with nine components altogether.

This process of merely adjoining further tensor components to those we already have does not, however, give us all those we require; in particular, there are some which vanish for every branch which lies in a plane, and which contain the determinants

$$E_{ijk} = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix}.$$

We define the principal t-invariants of the branch (14·1) in exactly the same way as those of the plane branch, using the dilating transformation

$$\left.\begin{aligned} x^{(1)} &= x &&= a_1 t + a_2 t^2 + a_3 t^3 + \dots, \\ y^{(1)} &= \frac{y}{x} - \frac{b_1}{a_1} &&= d_1 t + d_2 t^2 + d_3 t^3 + \dots, \\ z^{(1)} &= \frac{z}{x} - \frac{c_1}{a_1} &&= e_1 t + e_2 t^2 + e_3 t^3 + \dots \end{aligned}\right\} \tag{13·2}$$

and the dilating substitution of d_i for b_i and e_i for c_i; where $d_1, d_2, \dots$ are given by (4·3) with $D_{z(ij)}$ now written for D_{ij}, and $e_1, e_2, \dots$ by similar expressions with $-D_{y(ij)}$ in place of $D_{z(ij)}$. We define also
$$\Delta_{x(ij)} = d_i e_j - d_j e_i, \quad \Delta_{y(ij)} = e_i a_j - e_j a_i, \quad \Delta_{z(ij)} = a_i d_j - a_j d_i,$$
and it is clear that $\Delta_{y(1j)}$, $\Delta_{z(1j)}$ are given by (4·4), with $D_{y(1i)}$, $D_{z(1i)}$, respectively, in place of each D_{1i}; as, however,
$$D_{y(1i)} D_{z(1j)} - D_{z(1i)} D_{y(1j)} = a_1 E_{1ij},$$
we have
$$\begin{aligned} \Delta_{x(12)} &= E_{123}/a_1^3, \\ \Delta_{x(13)} &= (-a_2 E_{123} + a_1 E_{124})/a_1^4, \\ \Delta_{x(14)} &= \{(a_2^2 - a_1 a_3) E_{123} - a_1 a_2 E_{124} + a_1^2 E_{125}\}/a_1^5, \end{aligned}$$
....

Now taking $\mathbf{p}^\alpha = p_1^\alpha$ (i.e. $\mathbf{a} = a_1$, $\mathbf{b} = b_1$, $\mathbf{c} = c_1$) as the principal t-invariants of rank 1, and supposing those of rank $n-1$ to have been defined, we define those of rank n to be all components of all tensors of which any component arises as numerator on applying the dilating substitution to those of rank $n-1$. Thus as

$$b_1 \to D_{z(12)}/a_1^2, \quad c_1 \to -D_{y(12)}/a_1^2,$$

the three components of $\mathbf{D}_\alpha = D_{\alpha(12)}$ are the only principal t-invariants of rank 2. Similarly as

$$\mathbf{D}_x \to \frac{E_{123}}{a_1^3} = \frac{\mathbf{E}}{\mathbf{a}^3}, \quad \mathbf{D}_y \to \frac{\mathbf{G}_y^x}{\mathbf{a}^2}, \quad \mathbf{D}_z \to \frac{\mathbf{G}_z^x}{\mathbf{a}^2},$$

we see that $\mathbf{E} = E_{123}$ and the nine components of $\mathbf{G}_\alpha^\beta$ are the only principal t-invariants of rank 3. These satisfy of course a number of identities. In the first place, as

$$p_i^\alpha D_{\alpha(ij)} = p_j^\alpha D_{\alpha(ij)} = 0,$$

we have
$$\mathbf{p}^\alpha \mathbf{D}_\alpha = \mathbf{G}_\alpha^\alpha = \mathbf{p}^\alpha \mathbf{G}_\alpha^\beta = \mathbf{D}_\beta \mathbf{G}_\alpha^\beta = 0; \tag{13·3}$$

and further, if $\epsilon_{\alpha\beta\gamma}$, $\epsilon^{\alpha\beta\gamma}$ are the usual alternating tensors (equal to 1 or -1 if $\alpha\beta\gamma$ is an even or odd permutation of xyz, and to 0 otherwise)

$$\epsilon_{\alpha\beta\gamma}\mathbf{p}^\alpha \mathbf{G}_\delta^\beta = -2\mathbf{D}_\gamma \mathbf{D}_\delta, \quad \epsilon^{\alpha\beta\gamma}\mathbf{D}_\alpha \mathbf{G}_\beta^\delta = \mathbf{p}^\gamma \mathbf{p}^\delta \mathbf{E} \tag{13·4, 5}$$

of which the first, with $\gamma = \delta = z$, is (5·1).

Turning to the t-invariants of rank 4, we find that

$$\mathbf{G}_y^x \to \frac{\mathbf{I}_y^{xx}}{\mathbf{a}^2}, \quad \mathbf{G}_z^x \to \frac{\mathbf{I}_z^{xx}}{\mathbf{a}^2},$$

$$\mathbf{G}_y^y \to \frac{\mathbf{J}_{z,y}^{xx}}{\mathbf{a}^5}, \quad \mathbf{G}_y^z \to \frac{-\mathbf{J}_{y,y}^{xx}}{\mathbf{a}^5}, \quad \mathbf{G}_z^y \to \frac{\mathbf{J}_{z,z}^{xx}}{\mathbf{a}^5}, \quad \mathbf{G}_z^z \to \frac{-\mathbf{J}_{y,z}^{xx}}{\mathbf{a}^5},$$

where

$$\mathbf{I}_\gamma^{\alpha\beta} = (-p_1^\alpha p_3^\beta + 5p_2^\alpha p^\beta - p_3^\alpha p_1^\beta)\, D_{\gamma(12)} - \tfrac{3}{2}(p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta)\, D_{\gamma(13)} + p_1^\alpha p_1^\beta D_{\gamma(14)},$$

$$\begin{aligned}\mathbf{J}_{\gamma,\delta}^{\alpha\beta} &= -(p_1^\alpha p_3^\beta + 3p_2^\alpha p_2^\beta + p_3^\alpha p_1^\beta)\, D_{\gamma(12)} D_{\delta(12)} \\ &\quad + \tfrac{1}{2}(p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta)\,(4D_{\gamma(12)} D_{\delta(13)} + D_{\gamma(13)} D_{\delta(12)}) + p_1^\alpha p_1^\beta (D_{\gamma(14)} D_{\delta(12)} - 2D_{\gamma(13)} D_{\delta(13)}) \\ &= \mathbf{D}_\delta \mathbf{I}_\gamma^{\alpha\beta} - \mathbf{G}_\gamma^\alpha \mathbf{G}_\delta^\beta - \mathbf{G}_\delta^\alpha \mathbf{G}_\gamma^\beta\end{aligned}$$

are the full tensor forms of our former $\mathbf{I}$, $\mathbf{J}$; but also

$$\mathbf{G}_x^x \to \frac{\mathbf{S}^x}{a^3}, \quad \mathbf{G}_x^y \to \frac{\mathbf{T}_z^x}{a^6}, \quad \mathbf{G}_x^z \to -\frac{\mathbf{T}_y^x}{a^6}, \quad \mathbf{E} \to \frac{\mathbf{U}^{xx}}{a^4},$$

where

$$\mathbf{S}^\alpha = -3p_2^\alpha E_{123} + p_1^\alpha E_{124},$$

$$\mathbf{T}_\beta^\alpha = (p_2^\alpha D_{\beta(12)} - 2p_1^\alpha D_{\beta(13)})\, E_{123} + p_1^\alpha D_{\beta(123)} E_{124},$$

$$\mathbf{U}^{\alpha\beta} = (p_1^\alpha p_3^\beta + p_2^\alpha p_2^\beta + p_3^\alpha p_1^\beta)\, E_{123} - (p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta)\, E_{124} + p_1^\alpha p_1^\beta E_{134}$$

are t-invariant tensors of weights 8, 11, and 10, respectively. We note that those with two contravariant indices are symmetric, but $\mathbf{J}_{\gamma,\delta}^{\alpha\beta}$ is not symmetrical in γ, δ. (We shall insert a comma between indices to denote unsymmetry.) Thus $\mathbf{I}_\gamma^{\alpha\beta}$, $\mathbf{J}_{\gamma,\delta}^{\alpha\beta}$, $\mathbf{S}^\alpha$, $\mathbf{T}_\beta^\alpha$, $\mathbf{U}^{\alpha\beta}$ have respectively 18, 54, 3, 9, and 6 components, and these 90 are all the principal t-invariants of rank 4.

It is clear that just as the ratio $\mathbf{a}:\mathbf{b}$ determines the tangent P_0P_1 of the plane branch, so the ratios $\mathbf{a}:\mathbf{b}:\mathbf{c}$ determine that of the branch in S_3; similarly the ratios $\mathbf{D}_x:\mathbf{D}_y:\mathbf{D}_z$ determine the osculating plane $P_0P_1P_2$, and $\mathbf{D}_\alpha = 0$ is the condition for the branch to be inflected, i.e. for $P_0P_1P_2$ to be collinear. $\mathbf{E} = 0$ is the condition for $P_0P_1P_2P_3$ to be coplanar; similarly the vanishing of all the t-invariants of rank $\leqslant n$ containing the determinants E_{ijk} is the condition for $P_0 \ldots P_n$ to be coplanar, and the vanishing of all such t-invariants of all ranks is the condition for the whole branch to lie in a plane.

The extension of this theory to any number of dimensions is obvious. The principal t-invariants are defined by means of the standard dilating transformation in which all the co-ordinates except x are divided by x, and the origin moved to that of the transformed branch. What appear in S_3 as covariant tensors will in general have antisymmetric pairs of contravariant indices, or of course antisymmetric sets of $r-2$ covariant indices; thus the D_{ij} of the plane branch and $D_{\alpha(ij)}$ of that in S_3 appear in S_4 in either of the equivalent forms

$$D^{\alpha\beta}_{(ij)} = p^\alpha_i p^\beta_j - p^\alpha_j p^\beta_i, \quad D_{\alpha\beta(ij)} = \epsilon_{\alpha\beta\gamma\delta} p^\gamma_i p^\delta_j;$$

similarly the determinants E_{ijk} appear in S_4 as the covariant tensors

$$E_{\alpha(ijk)} = \epsilon_{\alpha\beta\gamma\delta} p^\beta_i p^\gamma_j p^\delta_k,$$

and we have a new series of determinants

$$F_{ijkl} = \epsilon_{\alpha\beta\gamma\delta} p^\alpha_i p^\beta_j p^\gamma_k p^\delta_l;$$

the components of $\mathbf{E}_\alpha = E_{\alpha(123)}$ are the co-ordinates of the 'osculating' S_3 $P_0P_1P_2P_3$, and the vanishing of all of them is the condition for $P_0P_1P_2P_3$ to be coplanar; similarly $\mathbf{F} = F_{1234} = 0$ is the condition for $P_0P_1P_2P_3P_4$ to be in an S_3.

And so on.

14. Parametrization of $W_{3,n}$

We can now proceed to the parametrization of $W_{3,n}$ in exactly the same way as for $W_{2,n}$. It is obvious in the first place that the plane $W_{3,1}$ has in it the natural co-ordinate system

$$X:Y:Z = \mathbf{a}:\mathbf{b}:\mathbf{c};$$

thus applying the dilating substitution we see that each plane $W_{3,1}(P_1)$ on $W_{3,2}$ can be parametrized by the monomials

$$a_1:d_1:e_1 = \mathbf{a}^4:\mathbf{a}\mathbf{D}_z:-\mathbf{a}\mathbf{D}_y,$$

and adjoining all components of these tensors we obtain the monomials

$$X^\alpha_\beta = \mathbf{p}^\alpha \mathbf{D}^\beta, \quad Y^{\alpha\beta\gamma\delta} = \mathbf{p}^\alpha\mathbf{p}^\beta\mathbf{p}^\gamma\mathbf{p}^\delta \tag{14·1}$$

which, allowing for differences of notation, is precisely Semple's parametrization of $W_{3,2}$. These co-ordinates satisfy

$$X^\alpha_\alpha = 0 \tag{14·2}$$

but are otherwise independent. If we take the direction coefficients of the line P_0P_1 (tangent to the branch) to be

$$1:\lambda:\mu = \mathbf{a}:\mathbf{b}:\mathbf{c}$$

the co-ordinates
$$X^x_y:X^x_z:Y^{xxxx} = -e_1:d_1:a_1$$

are independent (and serve as a co-ordinate system) in the generic $W_{3,1}(\mathsf{P}_1)$, and the rest are all determined from them by (14·2) and

$$\left.\begin{array}{ll} X^y_\beta = \lambda X^x_\beta, & X^z_\beta = \mu X^x_\beta, \\ Y^{\alpha\beta\gamma y} = \lambda Y^{\alpha\beta\gamma x}, & Y^{\alpha\beta\gamma z} = \mu Y^{\alpha\beta\gamma x}, \end{array}\right\} \tag{14·3}$$

which are accordingly the equations of the plane $W_{3,1}(\mathsf{P}_1)$.

Turning now to the parametrization of $W_{3,3}$, and applying the dilating substitution to (14·1), we have for the nine co-ordinate X^α_β

$$\left.\begin{array}{lll} a_1\Delta_{x(12)} = \dfrac{\mathbf{E}}{\mathbf{a}^2}, & a_1\Delta_{y(12)} = \dfrac{\mathbf{G}^x_y}{\mathbf{a}}, & a_1\Delta_{z(12)} = \dfrac{\mathbf{G}^x_z}{\mathbf{a}}, \\ d_1\Delta_{x(12)} = \dfrac{\mathbf{D}_z\mathbf{E}}{\mathbf{a}^5}, & d_1\Delta_{y(12)} = \dfrac{\mathbf{D}_z\mathbf{G}^x_y}{\mathbf{a}^4}, & d_1\Delta_{z(12)} = \dfrac{\mathbf{D}_z\mathbf{G}^x_z}{\mathbf{a}^4}, \\ e_1\Delta_{x(12)} = -\dfrac{\mathbf{D}_y\mathbf{E}}{\mathbf{a}^5}, & e_1\Delta_{(12)} = -\dfrac{\mathbf{D}_y\mathbf{G}_y}{\mathbf{a}^4}, & e_1\Delta_{z(12)} = -\dfrac{\mathbf{D}_y\mathbf{G}^x_z}{\mathbf{a}^4}, \end{array}\right\} \tag{14·4}$$

and for the fifteen co-ordinates $Y^{\alpha\beta\gamma\delta}$

$$\left.\begin{array}{ll} d_1^{4-i}e_1^i = \dfrac{(-1)^i}{\mathbf{a}^8}\mathbf{D}^i_y\mathbf{D}^{4-i}_z & (i = 0, 1, 2, 3, 4), \\ a_1 d_1^{3-i}e_1^i = \dfrac{(-1)^i}{\mathbf{a}^5}\mathbf{D}^i_y\mathbf{D}^{3-i}_z & (i = 0, 1, 2, 3), \\ a_1^2 d_1^{2-i}e_1^i = \dfrac{(-1)^i}{\mathbf{a}^2}\mathbf{D}^i_y\mathbf{D}^{2-i}_z & (i = 0, 1, 2), \\ a_1^3 d_1 = \mathbf{a}\mathbf{D}_z, \quad a_1^3 e_1 = -\mathbf{a}\mathbf{D}_y, \quad a_1^4 = \mathbf{a}^4. & \end{array}\right\} \tag{14·5}$$

Multiplying throughout by $\mathbf{a}^9$ and adjoining all components of each t-invariant tensor, we obtain the following set of monomials:

$$\left.\begin{array}{r} X^{\alpha_1\ldots\alpha_5,\beta}_{\gamma,\delta} = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_5}\mathbf{D}_\gamma\mathbf{G}^\beta_\delta, \\ X^{\alpha_1\ldots\alpha_8,\beta}_\delta = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_8}\mathbf{G}^\beta_\delta, \\ Y^{\alpha_1\ldots\alpha_4}_\gamma = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_4}\mathbf{D}_\gamma\mathbf{E}, \\ Y^{\alpha_1\ldots\alpha_7} = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_7}\mathbf{E}, \\ Z^\alpha_{\beta_1\ldots\beta_4} = \mathbf{p}^\alpha\mathbf{D}_{\beta_1}\ldots\mathbf{D}_{\beta_4}, \\ Z^{\alpha_1\ldots\alpha_4}_{\beta_1\beta_2\beta_3} = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_4}\mathbf{D}_{\beta_1}\mathbf{D}_{\beta_2}\mathbf{D}_{\beta_3}, \\ Z^{\alpha_1\ldots\alpha_7}_{\beta_1\beta_2} = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_7}\mathbf{D}_{\beta_1}\mathbf{D}_{\beta_2}, \\ Z^{\alpha_1\ldots\alpha_{10}}_\beta = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_{10}}\mathbf{D}_\beta, \\ Z^{\alpha_1\ldots\alpha_{13}} = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_{13}}. \end{array}\right\} \tag{14·6}$$

These 1569 expressions we take as homogeneous co-ordinates in S_{1568}, defining the generic point of an algebraic variety, which is our model of $W_{3,3}$. The proof that this is in fact a proper model of $W_{3,3}$ is typical of the similar proof for $W_{r,n}$, and just sufficiently complicated to contain all the points that need to be taken account of in the general case; we shall therefore give it in full, rather than struggle with the more complicated notation and longer explanations that would be necessary to a precise formulation of the proof in the general case.

The 1569 monomials (14·6) satisfy of course a lot of linear identities: first, by (13·3), all the three term relations obtained by contracting any covariant with any contravariant. index, namely

$$\left.\begin{aligned} &X^{\alpha_1\ldots\alpha_5,\gamma}_{\gamma,\delta} = X^{\alpha_1\ldots\alpha_5,\delta}_{\gamma,\delta} = X^{\alpha_1\ldots\alpha_4\gamma,\beta}_{\gamma,\delta} = X^{\alpha_1\ldots\alpha_4\delta,\beta}_{\gamma,\delta} = 0,\\ &X^{\alpha_1\ldots\alpha_8,\delta}_{\delta} = X^{\alpha_1\ldots\alpha_7\delta,\beta}_{\delta} = 0, \quad Y^{\alpha_1\alpha_2\alpha_3\gamma}_{\gamma} = 0,\\ &Z^{\gamma}_{\beta_1\beta_2\beta_3\gamma} = Z^{\alpha_1\alpha_2\alpha_3\gamma}_{\beta_1\beta_2\gamma} = Z^{\alpha_1\ldots\alpha_6\gamma}_{\beta\gamma} = Z^{\alpha_1\ldots\alpha_9\gamma}_{\gamma} = 0. \end{aligned}\right\} \tag{14·7}$$

Also by (13·4) we have

$$\epsilon_{\beta_1\gamma\delta} X^{\alpha_1\ldots\alpha_4,\delta}_{\beta_2,\beta_3} = Z^{\alpha_1\ldots\alpha_4}_{\beta_1\beta_2\beta_3}, \quad \epsilon_{\beta_1\gamma\delta} X^{\alpha_1\ldots\alpha_7\gamma,\delta}_{\beta_2} = Z^{\alpha_1\ldots\alpha_7}_{\beta_1\beta_2} \tag{14·8}$$

in which (7·6) are included; and by (13·5)

$$\epsilon^{\alpha_7\gamma\delta} X^{\alpha_1\ldots\alpha_5,\alpha_6}_{\gamma,\delta} = Y^{\alpha_1\ldots\alpha_5\alpha_6\alpha_7}. \tag{14·9}$$

There are 705 of these identities, reducing the ambient space of $W_{3,3}$ to S_{863}.

The monomials that are proportional to (14·4, 5) are those co-ordinates in which all the contravariant indices are x and all the covariant y or z, namely (in the same order as in (14·4, 5))

$$\left.\begin{matrix} Y^{x^7}, & X^{x^8}_{y}, & X^{x^8}_{z},\\ Y^{x^4}_{z}, & X^{x^5,x}_{z,y}, & X^{x^5,x}_{z,z},\\ -Y^{x^4}_{y}, & -X^{x^5,x}_{y,y}, & -X^{x^5,x}_{y,z} \end{matrix}\right\} \tag{14·10}$$

and

$$\left.\begin{aligned} &(-1)^i Z^{x}_{y^i z^{4-i}} \quad (i = 0, 1, 2, 3, 4),\\ &(-1)^i Z^{x^4}_{y^i z^{3-i}} \quad (i = 0, 1, 2, 3),\\ &(-1)^i Z^{x^7}_{y^i z^{2-i}} \quad (i = 0, 1, 2),\\ &Z^{x^{10}}_{z}, \quad -Z^{x^{10}}_{y}, \quad Z^{x^{13}}, \end{aligned}\right\} \tag{14·11}$$

where we have abbreviated to x^i, y^i, or z^i a row of i indices all equal to x, y, or z. The remaining co-ordinates are all determined linearly in terms of these by the identities (14·7, 8, 9) and the following equations of the ambient of $W_{3,2}(\mathsf{P}_1)$, in which $1:\lambda:\mu$ are as before the direction coefficients of the tangent $\mathsf{P}_0\mathsf{P}_1$ corresponding to the P_1 in question:

$$\left.\begin{aligned} X^{x^{5-i-j}y^iz^j,\beta}_{\gamma,\delta} &= \lambda^i\mu^j X^{x^5,\beta}_{\gamma,\delta},\\ X^{x^{8-i-j}y^iz^j,\beta}_{\delta} &= \lambda^i\mu^j X^{x^8,\beta}_{\delta},\\ Y^{x^{4-i-j}y^iz^j}_{\gamma} &= \lambda^i\mu^j Y^{x^4}_{\gamma},\\ Y^{x^{7-i-j}y^iz^j} &= \lambda^i\mu^j Y^{x^7};\\ Z^{y}_{\beta_1\ldots\beta_4} = \lambda Z^{x}_{\beta_1\ldots\beta_4}, \quad Z^{z}_{\beta_1\ldots\beta_4} &= \mu Z^{x}_{\beta_1\ldots\beta_4},\\ Z^{x^{4-i-j}y^iz^j}_{\beta_1\beta_2\beta_3} &= \lambda^i\mu^j Z^{x^4}_{\beta_1\beta_2\beta_3},\\ Z^{x^{7-i-j}y^iz^j}_{\beta_1\beta_2} &= \lambda^i\mu^j Z^{x^7}_{\beta_1\beta_2},\\ Z^{x^{10-i-j}y^iz^j}_{\beta} &= \lambda^i\mu^j Z^{x^{10}}_{\beta},\\ Z^{x^{13-i-j}y^iz^j} &= \lambda^i\mu^j Z^{x^{13}}. \end{aligned}\right\} \tag{14·12}$$

In fact, given the co-ordinates (14·10, 11), (14·12) determines those in which any of the contravariant indices α_i in (14·6) are y or z, (14·8) then determines those X's in which the index β is y or z, and finally (14·7) determines those in which any covariant index is x.

(14·10, 11) are all independent except that the sum of the three diagonal elements in (14·10) is zero; this is included in (14·9), and (14·4) shows that it arises by the dilating substitution from (14·2). Thus (14·12) are the equations of a subspace S_{22} in S_{863}; of the ∞^2 such subspaces given by all possible values of λ, μ, no two have a point in common, since every point has at least three non-zero co-ordinates proportional to $1:\lambda:\mu$.

Now it is clear that, the monomials (14·6) being isobaric (of weight 13), every free sequence $P_0 P_1 P_2 P_3$ determines a unique point of S_{863}, lying on the algebraic variety of which (14·6) is the parametrization, i.e. gives the co-ordinates of a generic point. As moreover the co-ordinates are all tensors, any linear transformation on (x, y, z) induces a self-collineation of the variety. Each of the subspaces (14·12) cuts this variety in a subvariety which, as (14·4, 5) were obtained from (14·1) by the dilating substitution, is a proper $W_{3,2}(P_1)$, i.e. a projective image of $W_{3,2}$, on which the sequences $P_1^{(1)} P_2^{(1)} P_3^{(1)}$ arising by dilation and having a common origin $P_1^{(1)}$ are mapped exactly as the sequences $P_0 P_1 P_2$ are mapped on $W_{3,2}$ itself. This argument does not apply directly of course when P_1 is in the plane $x = 0$, i.e. when the tangent to the branch lies in this plane; but the invariance under affine transformation shows that the $W_{3,2}(P_1)$ corresponding to such a P_1 is exactly like any other.

Every point of the variety lies in one and only one $W_{3,2}(P_1)$. Given therefore that our (i.e. Semple's) model of $W_{3,2}$ provides a one-one mapping of all sequences $P_0 P_1 P_2$ with the given origin P_0, including the unfree sequences, which for $n = 2$ are only those of type 2, the variety we have constructed provides a one-one mapping of all sequences $P_0 P_1 P_2 P_3$ with origin P_0, including the unfree sequences, since each of these gives rise on dilation to a sequence $P_1^{(1)} P_2^{(1)} P_3^{(1)}$ which is either free or of type 2, and has a well-defined image point on $W_{3,2}(P_1)$. (The actual parametrization of the unfree sequences will be dealt with in the next section.)

It is now clear that we can construct the model of $W_{3,n}$ recursively, exactly like that of $W_{2,n}$. Having obtained a parametrization of $W_{3,n-1}$, we apply the dilating substitution to all the t-invariant monomials entering into it; multiplication by $\mathbf{a}^{3^{n-1}}$ clears of fractions, and gives a set of t-invariant monomials, isobaric of weight $m_n = \frac{1}{2}(3^n - 1)$, in which the least exponent of $\mathbf{a}$ is unity; these are some components of certain tensors, those components namely in which every contravariant index is x and every covariant index y or z. Adjunction of the remaining components of these tensors completes the set of monomials which provide the parametrization of $W_{3,n}$, generated by ∞^2 projective images $\{W_{3,n-1}(P_1)\}$ of the model of $W_{3,n-1}$ from which we started, and with the property that every affine self-transformation of S_3 with P_0 as fixed point induces a self-collineation on $W_{3,n}$. Indeed, not only these affine transformations, but every self-transformation of S_3 with P_0 fixed and regular there induces a self-collineation on $W_{3,n}$; the proof is almost word for word the same as in § 9, and need not be repeated.

15. Parametrization of the Φ loci on $W_{3,2}$, $W_{3,3}$

The parametrization of $W_{3,n}$ we have obtained gives us of course a generic point, whose determinations are all the ordinary points of the variety, images of free sequences; and on these the group of self-collineations is transitive, exactly as in the case of $W_{2,n}$. As in that case also, the images of the unfree sequences are not determinations of the generic point we have obtained, and are not obtainable by allowing the coefficients in the expansion of the generic branch to tend to limits giving a singular branch; but must be obtained recursively from the mapping of sequences of lower species on $W_{3,n-1}(\mathsf{P}_1)$.

We shall, as in § 8, illustrate this by finding the image points on $W_{3,2}$, $W_{3,3}$ of all the unfree sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$, $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$. Of the former there is only type 2; of the latter, as well as the four types 2, 3, (23), and 2,3, considered for $r = 2$, we have also a type (23) 3, in which P_2 is indirectly proximate to P_0, and P_3 to both P_0, P_1; and we have also to distinguish in the case (23) between the general case, and that in which $\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$ (all proximate to P_0) are collinear; this we shall denote by $(2\bar{3})$. The distinction is significant, i.e. invariant under regular transformation, since this induces a linear transformation in the neighbourhood of P_0; the sequence of type $(2\bar{3})$ lies in a plane, and on every surface passing simply through P_0 and touching this plane, whereas the general sequence of type (23) lies on no surface simple at P_0; a cubic branch through the general sequence of type (23) is canonical in the sense defined by Cahit Arf, whereas one through a sequence of type $(2\bar{3})$ is not.

To avoid needless multiplication of diacritics, we shall, for each of the singular branches in question, denote by $\mathbf{p}'^{\alpha}$ (or $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$) the first triad a_i, b_i, c_i which are not all zero (these are in each case the direction coefficients of the tangent line of the branch); by $\mathbf{D}'_{\alpha}$ the first tensor $D_{\alpha(ij)}$ whose components are not all zero (these are similarly the co-ordinates of the osculating plane); and by $\mathbf{E}'$ the first non-zero determinant E_{ijk}.

For the sequence of type 2 we have to consider the ordinary cuspidal branch, for which

$$\mathbf{p}'^{\alpha} = p_2^{\alpha}, \quad \mathbf{D}'_{\beta} = D_{\beta(23)}, \quad \mathbf{E}' = E_{234};$$

and since in this case

$$\left.\begin{aligned} d_1 &= \frac{D_{z(23)}}{a_2^2}, \quad d_2 = \frac{-a_3 D_{z(23)} + a_2 D_{z(24)}}{a_2^3}, \quad \ldots, \\ e_1 &= -\frac{D_{y(23)}}{a_2^2}, \quad e_2 = \frac{a_3 D_{y(23)} - a_2 D_{y(24)}}{a_2^3}, \quad \ldots \end{aligned}\right\} \tag{15·1}$$

we have

$$\Delta_{x(12)} = \frac{\mathbf{E}'}{\mathbf{a}'^3}, \quad \Delta_{y(12)} = -\frac{\mathbf{D}'_y}{\mathbf{a}'}, \quad \Delta_{z(12)} = -\frac{\mathbf{D}'_z}{\mathbf{a}'}. \tag{15·2}$$

The parametrization of the plane $W_{3,1}(\mathsf{P}_1)$ gives

$$X_y^x : X_z^x : Y^{xxxx} = -e_1 : d_1 : a_1 = \mathbf{a}'\mathbf{D}'_y : \mathbf{a}'\mathbf{D}'_z : 0,$$

whence supplying the remaining co-ordinates from (14·2, 3), of course with

$$1 : \lambda : \mu = \mathbf{a}' : \mathbf{b}' : \mathbf{c}', \tag{15·3}$$

we have the following parametrization of the locus Φ_2 on $W_{3,2}$

$$X_{\beta}^{\alpha} = \mathbf{p}'^{\alpha}\mathbf{D}'_{\beta}, \quad Y^{\alpha\beta\gamma\delta} = 0. \tag{15·4}$$

As in the case of $W_{2,n}$, we consider also the loci such as $\Phi_{\bar{2}}$ of images of sequences satisfying conditions which are only invariant under projective, not under all regular, transformations in S_3. For $W_{3,2}\Phi_{\bar{2}}$, the locus of images of collinear sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$, is the only one of these that arises. As the condition for $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ to be collinear on a simple branch is $\mathbf{D}^{\beta} = 0$, we have merely to put this value into (14·1) to obtain the parametrization of the locus $\Phi_{\bar{2}}$ on $W_{3,2}$

$$X_{\beta}^{\alpha} = 0, \quad Y^{\alpha\beta\gamma\delta} = \mathbf{p}^{\alpha}\mathbf{p}^{\beta}\mathbf{p}^{\gamma}\mathbf{p}^{\delta}. \tag{15·5}$$

This is, from the point of view of the projective geometry of $W_{3,2}$, merely one member of a system of equivalence, on which the group of self-collineations of $W_{3,2}$ is transitive, and of which each member is the locus of images of sequences on the characteristic curves of a fixed net of surfaces with a simple base point at P_0.

For the parametrization of the locus Φ_2 on $W_{3,3}$, we have to put $a_1 = 0$ and the values of d_1, e_1, $\Delta_{\beta(12)}$ from (15·1, 2) in place of those in (14·4, 5); then, multiplying by $\mathbf{a}'^9$ we find that (14·10) and the top line of (14·11) are proportional to

$$\begin{matrix} 0, & 0, & 0, \\ \mathbf{a}'^4\mathbf{D}'_z\mathbf{E}', & -\mathbf{a}'^6\mathbf{D}'_y\mathbf{D}'_z, & -\mathbf{a}'^6\mathbf{D}'^2_z, \\ -\mathbf{a}'^4\mathbf{D}'_y\mathbf{E}', & \mathbf{a}'^6\mathbf{D}'^2_y, & \mathbf{a}'^6\mathbf{D}'_y\mathbf{D}'_z \end{matrix}$$

and
$$(-1)^i\mathbf{a}'\mathbf{D}'^i_y\mathbf{D}'^{4-i}_z \quad (i = 0, 1, 2, 3, 4),$$

the rest of (14·11) being all equal to zero. Finding the values of the remaining co-ordinates from (14·7, 8, 9, 12) we obtain the following parametrization of Φ_2

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_5,\alpha_6}_{\gamma,\delta} &= -\mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_5}\mathbf{p}'^{\alpha_6}\mathbf{D}'_\gamma\mathbf{D}'_\delta, \\ Y^{\alpha_1\ldots\alpha_4}_\gamma &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_4}\mathbf{D}'_\gamma\mathbf{E}', \\ Z^\alpha_{\beta_1\ldots\beta_4} &= \mathbf{p}'^{\alpha_1}\mathbf{D}'_{\beta_1}\mathbf{D}'_{\beta_2}\mathbf{D}'_{\beta_3}\mathbf{D}'_{\beta_4}, \\ &\text{all other co-ordinates} = 0. \end{aligned}\right\} \qquad (15\cdot6)$$

For the locus Φ_3 on $W_{3,3}$ we have to consider a (rhamphoid) cuspidal branch of the second species, with $a_1 = b_1 = c_1 = 0$, $D_{\beta(23)} = 0$, so that

$$\mathbf{p}'^\alpha = p^\alpha_2, \quad \mathbf{D}'_\beta = D_{\beta(24)}, \quad \mathbf{E}' = E_{245}.$$

For this branch

$$\left.\begin{aligned} d_1 &= 0, \quad d_2 = \frac{D_{z(24)}}{a_2^2}, \quad d_3 = \frac{-a_3 D_{z(24)} + a_2 D_{z(25)}}{a_2^3}, \quad \ldots, \\ e_1 &= 0, \quad e_2 = -\frac{D_{y(24)}}{a_2^2}, \quad e_3 = \frac{a_3 D_{y(24)} - a_3 D_{y(25)}}{a_2^3}, \quad \ldots, \end{aligned}\right\}$$

so that the dilated branch is cuspidal and

$$\Delta_{x(23)} = \frac{\mathbf{E}'}{\mathbf{a}'^3}, \quad \Delta_{y(23)} = \frac{\mathbf{G}''^x_y}{\mathbf{a}'^2}, \quad \Delta_{z(23)} = \frac{\mathbf{G}''^x_z}{\mathbf{a}'^2}.$$

where
$$\mathbf{G}''^\alpha_\beta = -2p^\alpha_3 D_{\beta(24)} + p^\alpha_2 D_{\beta(25)}$$

is a t-invariant of weight 9. We note that $\mathbf{p}'^\alpha\mathbf{G}''^\beta_\gamma - \mathbf{p}'^\beta\mathbf{G}''^\alpha_\gamma = 0$, and it is in fact convenient to write $\mathbf{G}''^\alpha_\beta = \mathbf{p}^\alpha\mathbf{D}''_\beta$, though of course $\mathbf{D}''_\beta$ is not a polynomial in the coefficients in (12·1); it is equal to $D_{\beta(25)} - 2\rho D_{\beta(24)}$, where $\rho = a_3/a_2 = b_3/b_2 = c_3/c_2$.

We have thus, to parametrize Φ_3, to substitute these values of d_2, e_2, $\Delta_{\beta(23)}$ in $a_2\Delta_{\beta(23)}$, $d_2\Delta_{\beta(23)}$, $e_2\Delta_{\beta(23)}$ for the three rows of (14·10), and equate all of (14·11) to zero, this being the application of the dilating transformation to (15·3); multiplying by $\mathbf{a}'^9$ we see that (14·10) are proportional to

$$\begin{matrix} \mathbf{a}'^7\mathbf{E}', & \mathbf{a}'^8\mathbf{G}''^x_y, & \mathbf{a}'^8\mathbf{G}''^x_z, \\ \mathbf{a}'^4\mathbf{D}'_z\mathbf{E}', & \mathbf{a}'^5\mathbf{D}'_z\mathbf{G}''^x_y, & \mathbf{a}'^5\mathbf{D}'_z\mathbf{G}''^x_z, \\ -\mathbf{a}'^4\mathbf{D}'_y\mathbf{E}', & -\mathbf{a}'^5\mathbf{D}'_y\mathbf{G}''^x_y, & -\mathbf{a}'^5\mathbf{D}'_y\mathbf{G}''^x_z \end{matrix}$$

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and again finding the remaining co-ordinates from (14·7, 8, 9, 12) we have the following parametrization of Φ_3

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_5,\beta}_{\gamma,\delta} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_5}\mathbf{D}'_\gamma\mathbf{G}''^{\beta}_{\delta} = \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_5}\mathbf{p}'^{\beta}\mathbf{D}'_\gamma\mathbf{D}''_\delta,\\ X^{\alpha_1\ldots\alpha_8,\beta}_{\delta} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_8}\mathbf{G}''^{\beta}_{\delta} = \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_8}\mathbf{p}'^{\beta}\mathbf{D}''_\delta,\\ Y^{\alpha_1\ldots\alpha_4}_{\gamma} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_4}\mathbf{D}'_\gamma\mathbf{E}',\\ Y^{\alpha_1\ldots\alpha_7} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_7}\mathbf{E}',\\ &\text{all other co-ordinates} = 0. \end{aligned}\right\} \tag{15·7}$$

The remaining Φ loci are similarly found; it is hardly necessary to go through all the workings. We have in fact

$\Phi_{(23)}$: cubic branch with

$$\mathbf{p}'^{\alpha} = p^{\alpha}_3, \quad \mathbf{D}'_\beta = D_{\beta(34)}, \quad \mathbf{E}' = E_{345};$$

the dilated branch is simple, giving

$$\left.\begin{aligned} Y^{\alpha_1\ldots\alpha_4}_{\beta} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_4}\mathbf{D}'_\beta\mathbf{E}',\\ Z^{\alpha}_{\beta_1\ldots\beta_4} &= \mathbf{p}'^{\alpha}\mathbf{D}'_{\beta}\ldots\mathbf{D}'_{\beta_4},\\ &\text{all other co-ordinates} = 0. \end{aligned}\right\} \tag{15·8}$$

$\Phi_{(2\bar{3})}$: the same but with $E_{345} = 0$ ($\mathbf{E}' = E_{346}$);

$$\left.\begin{aligned} Z^{\alpha}_{\beta_1\ldots\beta_4} &= \mathbf{p}'^{\alpha}\mathbf{D}'_{\beta_1}\ldots\mathbf{D}'_{\beta_4},\\ &\text{all other co-ordinates} = 0. \end{aligned}\right\} \tag{15·9}$$

$\Phi_{2,3}$: cubic branch with $D_{\beta(34)} = 0$;

$$\mathbf{p}'^{\alpha} = p^{\alpha}_3, \quad \mathbf{D}'_\beta = D_{\beta(35)}, \quad \mathbf{E}' = E_{356};$$

the dilated branch is cuspidal, giving

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_5,\alpha_6}_{\gamma,\delta} &= -\mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_6}\mathbf{D}'_\gamma\mathbf{D}'_\delta,\\ Y^{\alpha_1\ldots\alpha_4}_{\gamma} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_4}\mathbf{D}'_\gamma\mathbf{E}',\\ &\text{all other co-ordinates} = 0. \end{aligned}\right\} \tag{15·10}$$

$\Phi_{(23)3}$: quartic branch with $D_{\beta(45)} = 0$;

$$\mathbf{p}'^{\alpha} = p^{\alpha}_4, \quad \mathbf{D}'_\beta = D_{\beta(46)}, \quad \mathbf{E}' = E_{467};$$

the dilated branch is cuspidal, giving

$$\left.\begin{aligned} Y^{\alpha_1\ldots\alpha_4}_{\beta} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_4}\mathbf{D}'_\beta,\\ &\text{all other co-ordinates} = 0. \end{aligned}\right\} \tag{15·11}$$

Turning now to the Φ loci corresponding to conditions that are projectively but not regularly invariant, we have a primal $\Phi_{\bar{3}}$, locus of images of sequences in which P_3 lies in the osculating plane $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$. This is evidently obtained by putting $\mathbf{E} = 0$ in (14·6), so that its equations are $Y^{\cdots}_{\cdots} = 0$, the values of $X^{\cdots}_{\cdots}$, $Z^{\cdots}_{\cdots}$ being as in the generic point (the dots standing for any indices or none). This meets Φ_2, Φ_3, $\Phi_{(23)}$, $\Phi_{2,3}$ in the loci obtained by putting $\mathbf{E}' = 0$ (i.e. $Y^{\cdots}_{\cdots} = 0$) in (15·6, 7, 8, 10); it contains $\Phi_{2\bar{3}}$ and does not meet $\Phi_{(23)3}$, as we obviously expect.

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Lying on $\Phi_{\bar{3}}$ is the locus $\Phi_{\bar{2}}$ of images of sequences in which $P_0P_1P_2$ are collinear. As the conditions for this are $\mathbf{D}_\beta = 0$, from which also follows $\mathbf{E} = 0$, and

$$\mathbf{G}^\alpha_{\bar{\beta}} = p^\alpha_2\, D_{\beta(13)} = \mathbf{p}^\alpha \mathbf{D}'_\beta$$

say, the parametrization of $\Phi_{\bar{2}}$ is

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_8,\,\alpha_9}_{\delta} &= \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_9}\mathbf{D}'_\delta,\\ Z^{\alpha_1\ldots\alpha_{13}} &= \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_{13}},\\ \text{all other co-ordinates} &= 0.\end{aligned}\right\} \qquad (15\cdot12)$$

$\Phi_{\bar{2}\bar{3}}$ is given by $\mathbf{D}_\beta = \mathbf{E} = \mathbf{G}^\alpha_{\bar{\beta}} = 0$ in (14·6), i.e.

$$\left.\begin{aligned} Z^{\alpha_1\ldots\alpha_{13}} &= \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_{13}},\\ \text{all other co-ordinates} &= 0;\end{aligned}\right\} \qquad (15\cdot13)$$

and $\Phi_{\bar{2},3}$ refers to a cuspidal branch with $D_{\beta(23)} = D_{\beta(24)} = 0$,

$$\mathbf{p}'^\alpha = p^\alpha_{\bar{2}}, \quad \mathbf{D}'_\beta = D_{\beta(25)}, \quad \mathbf{E}' = E_{256};$$

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_8,\,\alpha_9}_{\delta} &= \mathbf{p}'^{\alpha_1}\ldots\mathbf{p}'^{\alpha_9}\mathbf{D}'_\delta,\\ \text{all other co-ordinates} &= 0.\end{aligned}\right\} \qquad (15\cdot14)$$

We note that as we expect $\Phi_{2,3}$, $\Phi_{(23)3}$, $\Phi_{\bar{2},3}$ are the intersections of Φ_2, $\Phi_{(23)}$, $\Phi_{\bar{2}}$ respectively with Φ_3.

16. Structure of $W_{3,n}$

We have seen that $W_{3,n}$ is $2n$-dimensional, and is generated by an ∞^2 congruence $\{W_{3,n-1}(P_1)\}$ of projective images of $W_{3,n-1}$, each of which is the locus of images of sequences having P_0P_1 in common, i.e. touching a fixed line P_0P_1 through P_0. These are of course the characteristic system of the net $|\Omega_1|$ of primals on $W_{3,n}$, each of which is the locus of images of sequences whose tangent P_0P_1 lies in a fixed plane through P_0. The equations of Ω_1 and $W_{3,n-1}(P_1)$ are clearly of the form $\mathbf{k}_\alpha\mathbf{p}^\alpha = 0$ and $\mathbf{p}^\alpha \equiv \mathbf{k}^\alpha$, where $\mathbf{k}^\alpha$, $\mathbf{k}_\alpha$ denote constants, and $\equiv$ means that the components of two tensors are proportional. Since $\{W_{3,n-1}(P_1)\}$ is a congruence of $(2n-2)$-folds on $W_{3,n}$, one of which passes through every point and no two of which have a point in common, and since $W_{3,1} = S_2$ is non-singular, it follows by an obvious induction that $W_{3,n}$ is non-singular; the argument is the same as for $W_{2,n}$, and applies equally to $W_{r,n}$, for all r.

$W_{3,n}$ is similarly generated by an ∞^4 congruence $\{W_{3,n-2}(P_2)\}$ of projective images of $W_{3,n-2}$, ..., and an ∞^{2n-2} congruence of planes $\{W_{3,1}(P_{n-1})\}$, each $W_{3,n-i}(P_i)$ being the locus of images of sequences $P_0\ldots P_n$ beginning with a particular subsequence $P_0\ldots P_i$. $W_{3,n}$ is thus a fibre space of $W_{3,n-i}$'s over $W_{3,i}$.

Every surface F passing simply through P_0 defines on $W_{3,n}$ a sequence $\Omega_1 \supset \Omega_2 \supset \ldots \supset \Omega_n$ of subvarieties, Ω_i being $(2n-i)$-dimensional, and being the locus of images of sequences in which $P_0\ldots P_i$ lie on F. In particular if F is a plane, we shall denote the corresponding Ω_i by $\bar{\Omega}_i$. It is clear that any Ω_i is generated by ∞^i members of the congruence $\{W_{3,n-i}(P_i)\}$, one corresponding to each sequence $P_0\ldots P_i$ on F, i.e. Ω_i is a fibre space of $W_{3,n-i}$'s over $W_{2,i}$. In particular Ω_n must be a birational model of $W_{2,n}$; it is in fact a projective image of $W_{2,n}$, as is easily seen by considering the $\bar{\Omega}_n$ corresponding to the plane $z = 0$; for this

$$c_1 = \ldots = c_n = 0$$

in (13·1), and in the parametrization of $W_{3,n}$ all the monomials vanish except those which provide the parametrization of $W_{2,n}$.

Ω_i traces on each of the ∞^1 $W_{3,n-1}(\mathsf{P}_1)$'s lying on the same Ω_1 the image of an Ω_{i-1} of $W_{3,n-1}$, which is the locus of images of sequences $\mathsf{P}_1^{(1)} \ldots \mathsf{P}_n^{(1)}$ in which $\mathsf{P}_1^{(1)} \ldots \mathsf{P}_i^{(1)}$ are on the transform $F^{(1)}$ of F.

If F, F' are two surfaces passing simply through P_0 and having contact of order i, i.e. such that their curve of intersection has an $(i+1)$-ple point at P_0, and $\mathsf{P}_0 \ldots \mathsf{P}_n$ is a free sequence, if $\mathsf{P}_0 \ldots \mathsf{P}_i$ are on F they are also on F'; thus the Ω_i's defined by F, F' are the same; if $\mathsf{P}_0 \ldots \mathsf{P}_{i+1}$ are on F they are also on F' if and only if $\mathsf{P}_0\mathsf{P}_1$ is one of the $i+1$ tangents to the curve of intersection of F, F'; thus the Ω_{i+1}'s defined by F, F' have in common their intersections with the $W_{3,n-1}(\mathsf{P}_1)$'s corresponding to these $i+1$ tangents. It follows that the Ω_{i-1}'s on Ω_i are a linear system compounded with a congruence of $(2n-i-2)$-folds, which on each member of the system form a rational pencil, the intersection of two members of the system consisting of $i+1$ members of the congruence, so that the projective model of the system is a rational ruled surface of order $i+1$, whose points correspond to the members of the congruence, and its generators to the pencil traced on Ω_i by $\{W_{3,n-1}(\mathsf{P}_1)\}$. The system $|\Omega_{i+1}|$ on Ω_i is thus of freedom $i+2$. (As well as its variable intersection, compounded with the congruence, it has also a base locus, as any two members of the system have the same intersection with Φ_j, $2 \leqslant j \leqslant i+1$; it is in fact easily verified that if $\mathsf{P}_0 \ldots \mathsf{P}_n$ is a sequence in which any of $\mathsf{P}_0 \ldots \mathsf{P}_{i+1}$ are satellites, and $\mathsf{P}_0 \ldots \mathsf{P}_{i+1}$ are on F, they are all also on F'.)

We have seen that $|\Omega_1|$ is a net of primals on $W_{3,n}$, whose characteristic system is the congruence $\{W_{3,n-1}(\mathsf{P}_1)\}$. The ∞^2 loci $\overline{\Omega}_2$ generate $W_{3,2}$ simply, one of them passing through each of its points, except that ∞^1 of them pass through each point of $\Phi_{\bar{2}}$, since every sequence $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ lies in a unique plane, unless it is collinear, when it lies in ∞^1 planes. Similarly, the ∞^2 loci $\overline{\Omega}_3$ generate $\Phi_{\tilde{3}}$ simply, except that ∞^1 of them pass through each point of $\Phi_{\overline{23}}$; and in general the ∞^2 loci $\overline{\Omega}_i$ generate the locus $\Phi_{\widetilde{3\ldots i}}$ of images of sequences in which $\mathsf{P}_0 \ldots \mathsf{P}_i$ are coplanar, simply except that ∞^1 of them pass through each point of $\Phi_{\overline{2\ldots i}}$.

It is clear that any net $|F|$ of surfaces in S_3 with a simple base point at P_0 defines a set of loci algebraically equivalent on $W_{3,n}$ to $\Phi_{\widetilde{3\ldots i}}$, $\Phi_{\overline{2\ldots i}}$, and the ∞^2 loci $\overline{\Omega}_i$, and transformed into these by the self-collineation of $W_{3,n}$ induced by the transformation in S_3, regular at P_0, which transforms the net $|F|$ into that of planes; they are defined in the same way, merely replacing the conditions that $\mathsf{P}_0 \ldots \mathsf{P}_i$ are coplanar or collinear by the conditions that these points lie on a surface of the net, or on a characteristic curve of the net.

Similarly given any pencil $|F|$ of surfaces in S_3, whose base curve passes simply through P_0, we can define a locus Σ_i of images of sequences such that $\mathsf{P}_0 \ldots \mathsf{P}_i$ are on some surface of the pencil. This is obviously of $2n-i+1$ dimensions, and generated by a pencil $|\Omega_i|$ corresponding to the individual surfaces of the pencil, and having in common the $W_{3,n-1}(\mathsf{P}_i)$ corresponding to the sequence $\mathsf{P}_0 \ldots \mathsf{P}_i$ on the base curve. If $|F|$ is a pencil of planes we shall denote by Σ_i by $\overline{\Sigma}_i$. Any Σ_1 is $W_{3,n}$ itself; $|\overline{\Sigma}_2|$ is a net of primals on $W_{3,n}$, with $\Phi_{\bar{2}}$ as base locus, and $\{\overline{\Omega}_2\}$ as characteristic system; in general $|\overline{\Sigma}_i|$ is a net of primals on $\Phi_{\widetilde{3\ldots i}}$ with $\Phi_{\overline{2\ldots i}}$ as base locus and $\{\overline{\Omega}_i\}$ as characteristic system. Obviously the Σ_i's corresponding to the ∞^2 pencils in one net (with a simple base point at P_0) are similarly related. We note that the equation of $\overline{\Sigma}_2$ is of the form $\mathbf{k}^\beta\mathbf{D}_\beta = 0$ as those of $\overline{\Omega}_2$ are $\mathbf{D}_\beta \equiv \mathbf{k}_\beta$.

17. Outline of geometry on $W_{3,2}$

We shall illustrate the remarks of the last section by examining some of the loci in question on $W_{3,2}$ and $W_{3,3}$. The former has been studied in detail by Semple, but it is convenient to recall here some of its geometrical properties, both in illustration of what has been said above, and because some of its subvarieties, as well as itself as a whole, will require to be to some extent known before we can embark on any examination of $W_{3,3}$.

The locus Φ_2 on $W_{3,2}$, parametrized in (15·4), is the sextic threefold in S_7, general (i.e. non-tangent) section of Segre's classical sextic fourfold in S_8, direct product of two planes in which $\mathbf{p}^\alpha$, $\mathbf{D}_\beta$ are homogeneous co-ordinate systems, the secant prime of course being given by (14·2). This threefold, which we shall encounter in several connexions in the course of this work, is familiar as $W^*_{2,1}$, the minimum order model of the aggregate of sequences $\mathsf{P}_0\mathsf{P}_1$ in a plane, i.e. of the figures of a point P_0 and a line $\mathsf{P}_0\mathsf{P}_1$ through it. It is equally of course the minimum model of the projectively equivalent aggregate of figures in S_3, of a line $\mathsf{P}_0\mathsf{P}_1$ through a fixed point P_0, and a plane $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ through the line. We thus expect to find this, or a birational model of it, as the locus of images on $W_{3,n}$ of sequences of any type such that the whole sequence is determined by the tangent $\mathsf{P}_0\mathsf{P}_1$ and osculating plane $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$, as is the case here.

This threefold is generated by two congruences $\{f\}$, $\{g\}$ of lines, with equations $\mathbf{p}^\alpha \equiv \mathbf{k}^\alpha$, $\mathbf{D}_\beta \equiv \mathbf{k}_\beta$, respectively, which are the characteristic systems of two nets $|F|$, $|G|$ of ruled cubics, with the equations $\mathbf{k}_\alpha \mathbf{p}^\alpha = 0$, $\mathbf{k}^\beta \mathbf{D}_\beta = 0$, generated by those lines of one system which meet a fixed line of the other (its directrix line). These satisfy

$$\left.\begin{aligned} F.F &= f, \quad F.G = f+g, \quad G.G = g; \\ F.g &= G.f = 1, \quad F.f = G.g = 0; \end{aligned}\right\} \tag{17·1}$$

thus on the projective model of the system $|iF+jG|$, parametrized

$$X^{\alpha_1\ldots\alpha_i}_{\beta_1\ldots\beta_j} = \mathbf{p}^{\alpha_1}\ldots\mathbf{p}^{\alpha_i}\mathbf{D}_{\beta_1}\ldots\mathbf{D}_{\beta_j}, \tag{17·2}$$

which we shall denote by $\mathfrak{w}^{(i,j)}$, so that the sextic from which we began is $\mathfrak{w}^{(1,1)}$, the images of f, g are curves of orders j, i, and those of F, G have prime sections which are images of $(i+j)f+jg$, $if+(i+j)g$, and orders $j(2i+j)$, $i(i+2j)$, and the order of $\mathfrak{w}^{(i,j)}$ is $3ij(i+j)$. We notice at once from their parametrizations that on $W_{3,3}$ the condition loci $\Phi_{(2\bar{3})}$, $\Phi_{(23)3}$, $\Phi_{2,3,\bar{3}}$, $\Phi_{\bar{2},3}$ are respectively $\mathfrak{w}^{(1,4)}$, $\mathfrak{w}^{(4,1)}$, $\mathfrak{w}^{(6,2)}$, $\mathfrak{w}^{(9,1)}$; and that in fact a sequence of either of these types is uniquely determined by its tangent and osculating plane, though in the last case this plane is not to be defined as $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$, these points being collinear, but as $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$.

The locus $\Phi_{\bar{2}}$ on $W_{3,2}$ is the surface of order 16 mapped on a plane by all quartic curves, and parametrized in (15·5); it has on it a homaloidal net of quartic curves $|a|$, images of the lines of the plane. $W_{3,2}$ is very simply generated by the congruence of planes $\{W\} = \{W_{3,1}(\mathsf{P}_1)\}$ joining each line f of Φ_2 to the corresponding point of $\Phi_{\bar{2}}$. Ω_1 is similarly generated by the ∞^1 planes that meet a given line g, i.e. that join the generators of a ruled cubic F to the corresponding points of a quartic curve a, and is thus a septimic threefold

Each surface Ω_2 is a ruled quintic, with one generator in each plane of the corresponding Ω_1, and with the same directrix line g, and is the residual section of Ω_1 by a prime through two of its generating planes; amongst these is $\overline{\Omega}_2$, generated by the lines joining corresponding points of g, a. $\overline{\Sigma}_2$ is generated by ∞^1 ruled quintics $\overline{\Omega}_2$ whose directrix lines g meet a fixed f, and whose quartic directrices a form a pencil on $\Phi_{\bar{2}}$ through a point O; it contains just one plane $W = Of$, meeting each of the pencil $|\overline{\Omega}_2|$ in a line. The tangent planes to $|\overline{\Omega}_2|$ at O evidently generate a quadric cone, so that O is a double point of $\overline{\Sigma}_2$.

A base for primals on $W_{3,2}$ consists of $|\Omega_1|$, Φ_2; one for surfaces of $\{W\}$, $|F|$, $|G|$; and one for curves of $\{f\}$, which is ∞^4 as it includes any line in a plane W, and $\{g\}$. The intersection table is as follows (Π denoting the prime sections)

	Ω_1	Φ_2	Π
Ω_1	W	F	$4W+F$
Φ_2	F	$-3F+G$	$F+G$
W	0	f	f
F	f	$-2f+g$	$2f+g$
G	$f+g$	$-3f-2g$	$f+2g$
f	1	-3	1
g	0	1	1

	W	F	G
W	0	0	1
F	0	1	-2
G	1	-2	-3

and there are the following equivalences

$$\Pi = 4\Omega_1 + \Phi_2, \quad \Sigma_2 = 3\Omega_1 + \Phi_2, \tag{17·3}$$

$$\Omega_2 = 2W + F, \quad \Phi_{\bar{2}} = 7W + 2F + G, \tag{17·4}$$

$$a = 3f + g. \tag{17·5}$$

18. Outline of geometry on $W_{3,3}$

For the detailed discussion of $W_{3,3}$ we shall (apart from the Φ loci and other condition loci Ω_i, Σ_i, for which finding new names would be otiose and confusing) denote fourfolds and threefolds by capital and small German letters, and surfaces and curves by capital and small italics. In particular we shall use $\mathfrak{W}$, W for $W_{3,2}(\mathsf{P}_1)$ and the plane $W_{3,1}(\mathsf{P}_2)$.

Since the congruence $\{W\}$ is mapped on $W_{3,2}$ (which is what we mean by describing $W_{3,3}$ as a fibre space of planes over $W_{3,2}$), every subvariety of $W_{3,2}$ has on $W_{3,3}$, as well as an obvious image (to within equivalence) on each $\mathfrak{W}$, an image in $\{W\}$, namely, the variety (of two more dimensions than itself) generated by the corresponding subsystem of $\{W\}$. The images in $\{W\}$ of the condition loci Φ_2, $\Phi_{\bar{2}}$, Ω_1, Ω_2, Σ_2 on $W_{3,2}$ are the similarly named condition loci on $W_{3,3}$, since the conditions only involve $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$, and every sequence $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ satisfying them determines a point of $W_{3,2}$ and a plane W on $W_{3,3}$; similarly the image of W is $\mathfrak{W}$; those of F, G, f, g, a we shall denote by $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{a}$.

The image on $\mathfrak{W}$ of Φ_2 on $W_{3,2}$ is a $\mathfrak{w}^{(1,1)}$ which we shall simply call $\mathfrak{w}$, and is the trace of Φ_3 on $\mathfrak{W}$. That of Ω_1 on the other hand is a threefold generated by ∞^1 planes W, locus of images of sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$ in which $\mathsf{P}_0\mathsf{P}_1$ is a fixed line and $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ a fixed plane through it; as the locus of images on $W_{3,2}$ of sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ with this property is a line f, this septimic threefold is what we have already defined as $\mathfrak{f}$. The images on $\mathfrak{W}$ of the ruled cubics F, G, and the lines f, g will be denoted by the same symbols F, G, f, g. Thus $\{f\}$ is an ∞^6 line system consisting of all lines in all planes $\{W\}$, $\{g\}$ is an ∞^4 congruence simply generating Φ_3, ∞^2 of them on each $\mathfrak{w}$. As Φ_3 meets every plane W in a line f', $\{f'\}$ is also an ∞^4 line congruence simply generating Φ_3, and every subvariety of $W_{3,2}$ has a further image in $\{f'\}$, namely, the variety (of one more dimension than itself) on Φ_3 generated by the corresponding subsystem of $\{f'\}$; and this is simply the trace on Φ_3 of its image in $\{W\}$.

There are as we have seen other images of $\mathfrak{w}$ on $W_{3,3}$, since $\Phi_{(2\bar{3})}$, $\Phi_{(23)3}$, $\Phi_{2,3,\bar{3}}$, $\Phi_{\bar{2},3}$, parametrized in (15·9), (15·11), (15·10) (with $\mathbf{E}' = 0$), (15·14), are respectively $\mathfrak{w}^{(1,4)}$; $\mathfrak{w}^{(4,1)}$, $\mathfrak{w}^{(6,2)}$, $\mathfrak{w}^{(9,1)}$. On each of these we shall denote the images of F, G, f, g, loci of images of sequences of the type in question with respectively tangent in a fixed plane, osculating plane through a fixed line, fixed tangent, and fixed osculating plane, by M, N, m, n, with in each case the suffixes of the corresponding Φ locus. Each of these being parametrized in terms of the co-ordinates $\mathbf{p}'^{\alpha}$, $\mathbf{D}'_{\beta}$ of the tangent and osculating plane have a natural mapping on each other; and it is clear also from (15·8), (15·10), (15·6) (with $\mathbf{E}' = 0$), and (15·7) (where $\mathbf{E}' = 0$ makes $\mathbf{D}'_{\beta} \equiv \mathbf{D}''_{\beta}$), that $\Phi_{(23)}$, $\Phi_{2,3}$, $\Phi_{2,\bar{3}}$, and $\Phi_{3,\bar{3}}$ are each generated by ∞^3 lines joining corresponding points of two of these $w^{(i,j)}$'s, namely $\Phi_{(2\bar{3})}$ and $\Phi_{(23)3}$, $\Phi_{(23)3}$ and $\Phi_{2,3,\bar{3}}$, $\Phi_{(2\bar{3})}$ and $\Phi_{2,3,\bar{3}}$, $\Phi_{2,3,\bar{3}}$ and $\Phi_{\bar{2},3}$. Finally, from (15·6), Φ_2 is generated by

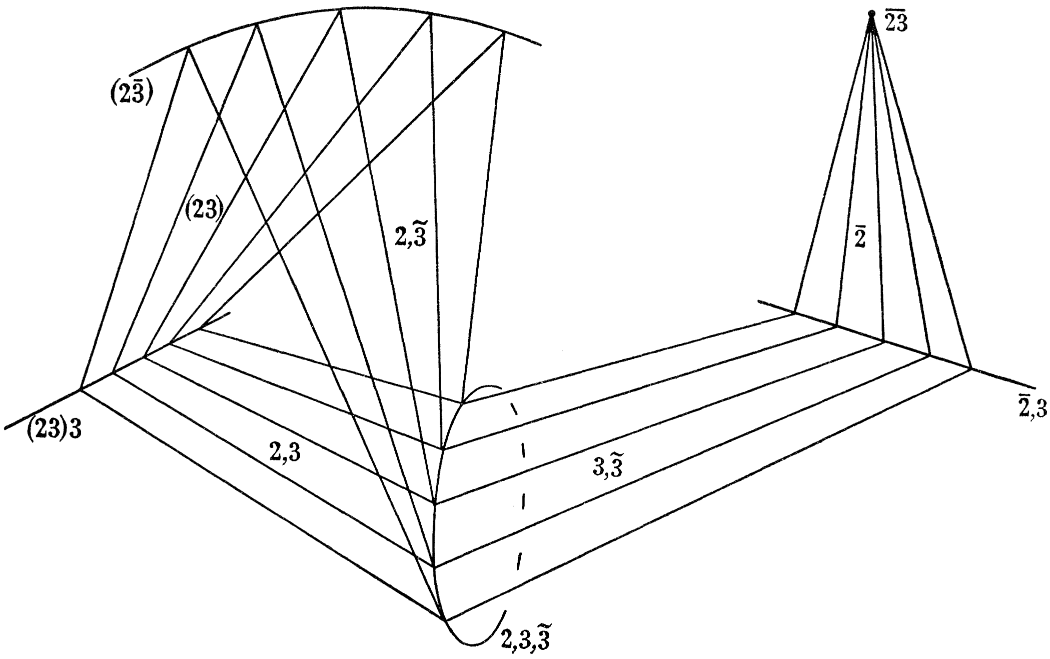

FIGURE 5. Special loci on $W_{3,3}$ (traces on $W'_{3,2}(\mathsf{P}_1)$).

∞^3 planes joining corresponding points of $\Phi_{(2\bar{3})}$, $\Phi_{(23)3}$, $\Phi_{2,3,\bar{3}}$. On each of these fourfolds we shall denote the threefolds generated by lines meeting a surface M, N on each of the directrix $\mathfrak{w}^{(i,j)}$'s by $\mathfrak{s}$, $\mathfrak{t}$, and the surfaces generated by lines meeting a curve m, n, by S, T; the similarly generated fourfolds and threefolds on Φ_2 are what we have already denoted by $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{f}$, $\mathfrak{g}$. The remaining fourfold Φ locus, $\Phi_{\bar{2}}$, has a generation different from the others, and more analogous to $W_{3,2}$, namely by ∞^2 planes W, each joining a line $m_{\bar{2},3}$ of $\Phi_{\bar{2},3}$ to the corresponding point of the surface $\Phi_{\bar{2}\bar{3}}$, which is of order 169, being mapped on a plane by all curves of order 13, and having on it a homaloidal net $|e|$ of curves of order 13. ($\Phi_{\bar{2}}$ is not, however, an unexceptional birational model of $W'_{3,2}$; it would be so only if the orders of the curves e, $n_{\bar{2},3}$ differed by 3 instead of 4 as is the case.)

On $\mathfrak{W}$, corresponding to a fixed tangent, Φ_2 traces a threefold $\mathfrak{f}$ and Φ_3 a threefold $\mathfrak{w}$, $\Phi_{\bar{2}}$ a plane W and each of the other fourfold Φ loci a ruled surface S, each of the threefold Φ loci a curve m, and $\Phi_{\bar{2}\bar{3}}$ a single point. These are shown in figure 5. Since $m_{(2\bar{3})}$, $m_{(23)3}$, $m_{2,3,\bar{3}}$, $m_{\bar{2},3}$ are of orders 4, 1, 2, 1, the ruled surfaces $S_{(23)}$, $S_{2,3}$, $S_{2,\bar{3}}$, $S_{3,\bar{3}}$ are of orders 5, 3, 6, 3, respectively; in particular as the two cubics $S_{2,3}$, $S_{3,\bar{3}}$, meeting in the conic $m_{2,3,\bar{3}}$ are both on the $\mathfrak{w}$ traced by Φ_3, and as the generators of the former and the directrix line of the latter are seen to lie in planes W, the former is a surface F, and the latter G. It follows that whereas the generators of $\Phi_{(2\bar{3})}$, $\Phi_{2,3}$, $\Phi_{2,\bar{3}}$ belong to the system $\{f\}$, those of $\Phi_{3,\bar{3}}$ belong to $\{g\}$, as do also the generators $m_{(23)3}$ of $\Phi_{(23)3}$.

In order to exhibit similarly the curves n on the threefold and ruled surfaces T on the fourfold Φ loci, we have to consider the locus $\overline{\Omega}_2$, which corresponds to a fixed osculating plane in the same way as $\mathfrak{W}$ to a fixed tangent. We have seen that $\overline{\Omega}_3$, for the plane $z = 0$, can be obtained by putting $c_1 = c_2 = c_3 = 0$ in (13·1); when the resulting values of the t-invariants are substituted in (14·6) all the co-ordinates vanish except those in which every covariant index is z and every contravariant index x or y, and these are just the monomials

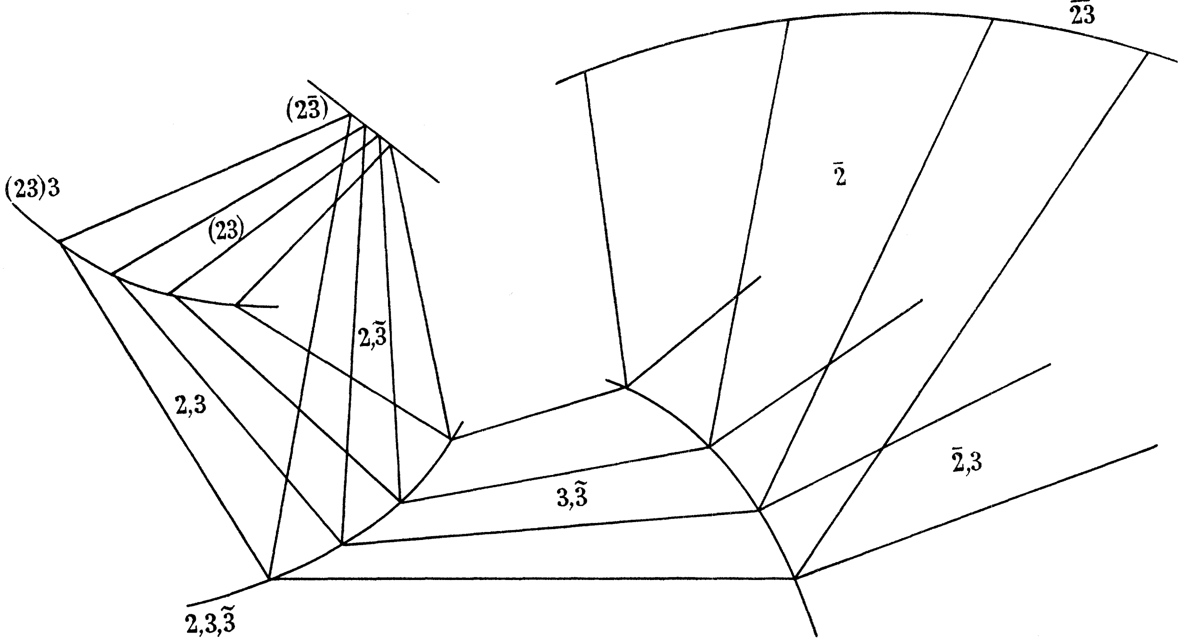

FIGURE 6. Special loci on $W_{3,3}$ (traces on Ω_2).

(7·5) which parametrize $W_{2,3}$, of which $\overline{\Omega}_3$ is a projective image. To obtain $\overline{\Omega}_2$ we only put $c_1 = c_2 = 0$, when as well as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{D}_z$, $\mathbf{G}_z^x$, $\mathbf{G}_z^y$ we have the following non-vanishing t-invariants

$$\mathbf{G}_x^x = -\mathbf{ab}c_3, \quad \mathbf{G}_x^y = -\mathbf{b}^2c_3, \quad \mathbf{G}_y^x = \mathbf{a}^2c_3, \quad \mathbf{G}_y^y = \mathbf{ab}c_3, \quad \mathbf{E} = \mathbf{D}_zc_3,$$

and the monomials in (14·6) which contain these and do not vanish reduce to

$$\left.\begin{array}{ll} \mathbf{a}^{4-i}\mathbf{b}^i\mathbf{D}_z^2c_3 & (i = 0, \ldots, 4), \\ \mathbf{a}^{7-i}\mathbf{b}^i\mathbf{D}_zc_3 & (i = 0, \ldots, 7), \\ \mathbf{a}^{10-i}\mathbf{b}^ic_3 & (i = 0, \ldots, 10). \end{array}\right\} \qquad (18\cdot1)$$

These parametrize a birational model B^{17} of the ruled quintic $W_{2,2}$, on which the directrix appears as a quartic curve and the generators as conics; and $\overline{\Omega}_2$ is generated by ∞^2 planes W, each joining a generator f of $\overline{\Omega}_3$ (image of l_3 on $W_{2,3}$) to the corresponding point of B^{17}.

We see in figure 6 the traces of the threefold and fourfold Φ loci on $\overline{\Omega}_2$, more or less in corresponding positions to figure 5. As the curves $n_{(2\bar{3})}$, $n_{(23)3}$, $n_{2,3,\bar{3}}$, $n_{\bar{2},3}$ are of orders 1, 4, 6, 9, the ruled surfaces $T_{(23)}$, $T_{2,3}$, $T_{2,\bar{3}}$, $T_{3,\bar{3}}$ are of orders 5, 10, 7, 15; and $\mathfrak{g}$, generated by planes joining corresponding points of the first three n curves, is of order 11. The lines $n_{(2\bar{3})}$ do not belong to the system $\{f\}$, since each does not lie in a plane W but is unisecant to the generating planes of $\mathfrak{g}$; nor do they belong to $\{g\}$, as they are not on Φ_3; they therefore form a third line system on $W_{3,3}$, which we shall denote by $\{h\}$, ∞^2 and generating $\Phi_{(2\bar{3})}$; and just as $\mathfrak{f}$ is generated by the planes W that meet a fixed line g, $\mathfrak{g}$ is generated by those that meet a fixed line h.

Since if $P_0 P_1 P_2$ are collinear and P_1 is in a given plane through P_0, P_2 is in this plane likewise, $\Phi_{\bar{2}}$ traces on $\bar{\Omega}_2$ not merely a surface but a threefold $\mathfrak{a}$ (image in $\{W\}$ of a curve a on $W_{3,2}$, which is likewise the trace of $\Phi_{\bar{2}}$ on $\bar{\Omega}_2$) meeting $\Phi_{\bar{2}\bar{3}}$ in a curve e. Similarly $\Phi_{\bar{2},3}$ traces on $\bar{\Omega}_2$ not only the nonic curve $n_{\bar{2},3}$, but the surface $M_{\bar{2},3}$ of order 19, generated by the lines $m_{\bar{2},3}$ that meet $n_{\bar{2},3}$. The threefold $\mathfrak{a}$ is generated by the planes W joining each generator $m_{\bar{2},3}$ of $M_{\bar{2},3}$ to the corresponding point of e, so that the order of $\mathfrak{a}$ is 32. We see also in the figure a ruled surface E of order 22, generated by lines (one in each generating plane of $\mathfrak{a}$) joining corresponding points of $n_{\bar{2},3}$, e. This is the locus of images of sequences of type $\bar{2}$ (i.e. with $P_0 P_1 P_2$ collinear) that are wholly in the plane defining $\bar{\Omega}_2$.

Since if $P_0 P_1 P_2$ are in a given plane and $P_0 P_1 P_2 P_3$ are co-planar, either P_3 is also in the given plane, or $P_0 P_1 P_2$ are collinear, the trace of $\Phi_{\bar{3}}$ on $\bar{\Omega}_2$ breaks up into $\bar{\Omega}_3$ and $\mathfrak{a}$, meeting in the surface E. On $\bar{\Omega}_3$, the images of the surfaces Φ_2, Φ_3, $\Phi_{\bar{2}}$ on $W_{2,3}$ are $T_{2,\bar{3}}$, $T_{3,\bar{3}}$, and E; in fact, this part of figure 6 is readily identified with figure 1. $\bar{\Omega}_3$ is thus generated by a pencil $|K|$ of ruled quintics (image of $|W_{2,2}(P_1)|$ on $W_{2,3}$) each having a generator of $T_{3,\bar{3}}$ as directrix line, and the generators of $T_{2,\bar{3}}$, E that meet this as two of its generators.

Still taking the plane defining $\bar{\Omega}_2$ to be $z = 0$, we can find the trace of Φ_3 on $\bar{\Omega}_2$ by putting $c_2 = c_3 = c_4 = 0$ in (15·7). This makes $\mathbf{c}' = \mathbf{D}'_x = \mathbf{D}'_y = 0$ and

$$\mathbf{D}''_x = -\mathbf{b}'c_5, \quad \mathbf{D}''_y = \mathbf{a}'c_5, \quad \mathbf{E}' = \mathbf{D}_z c_5;$$

the monomials that do not vanish in (15·7) are

$$\mathbf{a}'^{6-i}\mathbf{b}'^{i}\mathbf{D}'_z\mathbf{D}''_z \quad (i = 0, \ldots, 6), \qquad \mathbf{a}'^{9-i}\mathbf{b}'^{i}\mathbf{D}''_z \quad (i = 0, \ldots, 9),$$

which parametrize $T_{3,\bar{3}}$, and those in (18·1) with $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{D}'_z$, c_5 in place of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{D}_z$, c_3, the co-ordinates proportional to these being the same as before. Thus the trace of Φ_3 on $\bar{\Omega}_2$ is generated by ∞^2 lines f', joining corresponding points of $T_{3,\bar{3}}$, B^{17}, both of which are birational images of $W_{2,2}$; the lines that meet $n_{2,3,\bar{3}}$ generate $T_{2,3}$ ($n_{(23)3}$ being as we have said the directrix curve on B^{17}), and those that meet a generator of $T_{3,\bar{3}}$ trace a conic on B^{17} and generate a ruled cubic F.

$\mathfrak{W}$ and $\bar{\Omega}_2$ have no intersection in general, but if the line $P_0 P_1$ defining $\mathfrak{W}$ lies in the plane $P_0 P_1 P_2$ defining $\bar{\Omega}_2$ they have in common a threefold $\mathfrak{f}$. Such threefolds appear partially on both figures 5 and 6, being generated by the planes W that meet a line g on $\Phi_{3,\bar{3}}$; for each such line g in each figure two generating planes of the corresponding $\mathfrak{f}$ appear, one on Φ_2 and one on $\Phi_{\bar{2}}$; only in figure 5 the second of these planes is the same for all the generators of $S_{3,\bar{3}}$. The ruled quintics K, one on each of these $\mathfrak{f}$'s, generate the trace of $\Phi_{\bar{3}}$ in figure 5 as well as in figure 6, and they appear in the same way in the two figures, the directrix showing, and two generators, one on $\Phi_{2,\bar{3}}$ and one on $\Phi_{\bar{2}}$; but in figure 5 these second generators form a pencil in the plane trace of $\Phi_{\bar{2}}$; the trace of $\Phi_{\bar{3}}$ on W is in fact the image on W of a Σ_2 on $W_{3,2}$, and has a double point at the point trace of $\Phi_{\bar{2}\bar{3}}$. $\Phi_{\bar{2}\bar{3}}$ is accordingly a locus of double points on $\Phi_{\bar{3}}$.

19. Base and intersection theory on $W_{3,3}$

Before seeking for a base, it is as well to obtain some equivalences between the numerous curves, surfaces, and threefolds which we have defined. By considering prime sections of the ruled surfaces in figures 5 and 6, and of the threefolds $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{a}$, we have

$$\left.\begin{array}{ll} m_{(2\bar{3})} = 3f+g, & n_{(2\bar{3})} = h, \\ m_{(23)3} = g, & n_{(23)3} = 3f+h, \\ m_{2,3,\bar{3}} = f+g, & n_{2,3,\bar{3}} = 5f+h, \\ m_{\bar{2},3} = f, & n_{\bar{2},3} = 5f+3g+h; \end{array}\right\} \tag{19·1}$$

$$\left.\begin{array}{ll} K = S_{(23)} = 2W+F, & T_{(23)} = R, \\ S_{2,3} = F, & T_{2,3} = 5W+R, \\ S_{2,\bar{3}} = 3W+F, & T_{2,\bar{3}} = 2W+R, \end{array}\right\} \tag{19·2}$$

$$e = 9f+3g+h, \quad E = 3W+M_{\bar{2},3}, \tag{19·3}$$

where we have defined R as the lowest in order of the T surfaces (it is in fact a quintic). On the other hand as the images on $\bar{\Omega}_2$ of Φ_2, $\Phi_{\bar{2}}$ on $W_{2,3}$ are $T_{2,\bar{3}}$, E, we have from (12·1)

$$E = 3K+T_{2,\bar{3}} = 8W+3F+R,$$

and from (19·3)

$$M_{\bar{2},3} = 5W+3F+R. \tag{19·4}$$

Now if Φ is any of the four Φ loci generated by lines joining corresponding points on two of the threefolds $\mathfrak{w}^{(i,j)}$, $\mathfrak{w}^{(i',j')}$, distinguishing the surfaces M, N and curves m, n on the latter by dashes, we see by considering primes through $\mathfrak{w}^{(i,j)}$, $\mathfrak{w}^{(i',j')}$ that

$$\left.\begin{array}{l} \Pi\,.\,\Phi = \mathfrak{w}^{(i,j)}+i'\mathfrak{s}+j'\mathfrak{t} \quad = \mathfrak{w}^{(i',j')}+i\mathfrak{s}+j\mathfrak{t}, \\ \Pi\,.\,\mathfrak{s} = M+(i'+j')\,S+j'T = M'+(i+j)\,S+jT, \\ \Pi\,.\,\mathfrak{t} = N+i'S+(i'+j')\,T \quad = N'+iS+(i+j)\,T, \end{array}\right\} \tag{19·5}$$

where Π as usual denotes a prime section. Thus defining $I = M_{(23)3}$, $J = N_{(2\bar{3})}$, which are the surfaces M, N of lowest order, both nonic and ruled in $\{g\}$, $\{h\}$, respectively, and putting in the values of the various surfaces S, T from (19·2) and of $M_{\bar{2},3}$ from (19·4) we find

$$\left.\begin{array}{lll} M_{(2\bar{3})} = 3R+I, & M_{(23)3} = I, & M_{2,3,\bar{3}} = 5W+3F+R+I, \\ N_{(2\bar{3})} = J, & N_{(23)3} = 6W+3F+J, & N_{2,3,\bar{3}} = 21W+5F+3R+J, \\ S_{3,\bar{3}} = G, & T_{3,\bar{3}} = 2G+I, & N_{\bar{2},3} = 21W+5F+7G+3R+2I+J. \end{array}\right\} \tag{19·6}$$

Further, considering primes through the two directrix surfaces $T_{3,\bar{3}}$, B^{17} of the threefold intersection of $\bar{\Omega}_2$ with Φ_3 we have

$$\Pi\,.\,\Omega_2\,.\,\Phi_3 = T_{3,\bar{3}}+2F+10K = B^{17}+F+9K,$$

so that

$$B^{17} = T_{3,\bar{3}}+K-F = 2W+2G+I. \tag{19·7}$$

This gives us, as linear combinations of W, F, G, R, I, J, all the surfaces on $W_{3,3}$ which have been considered, with the exception of $\Phi_{\bar{2}\bar{3}}$. For this, we have to consider intersection relations on $\Phi_{\bar{2}}$, which as we have seen is analogous (though not identical) in its structure to $W_{3,2}$. A base on $\Phi_{\bar{2}}$ consists of $\mathfrak{a}$, $\Phi_{\bar{2},3}$, W, $M_{\bar{2},3}$, $N_{\bar{2},3}$, $m_{\bar{2},3} = f$, $n_{\bar{2},3}$; and as the prime sections and intersections with $\mathfrak{a}$ (on $\Phi_{\bar{2}}$) of each of these three surfaces are easily found, we can construct the intersection table for threefolds and surfaces on $\Phi_{\bar{2}}$ as follows, the middle column being obtained from the identity between the three columns that follows from the fact that the prime sections of $\Phi_{\bar{2}}$ are the system $13\mathfrak{a}+\Phi_{\bar{2},3}$

	$\mathfrak{a}$	$\Phi_{\bar{2},3}$	$13\mathfrak{a}+\Phi_{\bar{2},3}$
W	0	f	f
$M_{\bar{2},3}$	f	$-3f+n_{\bar{2},3}$	$10f+n_{\bar{2},3}$
$N_{\bar{2},3}$	$f+n_{\bar{2},3}$	$-4f-3n_{\bar{2},3}$	$9f+10n_{\bar{2},3}$

Since $\Phi_{\bar{2}\bar{3}}$ does not meet $\Phi_{\bar{2},3}$, and traces on B^{17} a curve e, or $4f+n_{(2\bar{3})}$, it follows that

$$\Phi_{\bar{2}} = 13W+3M_{\bar{2},3}+N_{\bar{2},3} = 49W+14F+7G+6R+2I+J. \tag{19·8}$$

Some relations between the various threefold and fourfolds can be obtained by considering the sections of Φ_2 by primes through $\Phi_{(23)}$, $\Phi_{2,3}$, and $\Phi_{2,3,3}$; we have in fact evidently

$$\left.\begin{aligned} \Pi\,.\,\Phi_2 &= \Phi_{(23)}+6\mathfrak{F}+2\mathfrak{G} = \Phi_{2,3}+\mathfrak{F}+4\mathfrak{G} = \Phi_{2,\bar{3}}+4\mathfrak{F}+\mathfrak{G},\\ \Pi\,.\,\mathfrak{F} &= \mathfrak{s}_{(23)}+8\mathfrak{f}+2\mathfrak{g} \quad = \mathfrak{s}_{2,3}+5\mathfrak{f}+4\mathfrak{g} \quad = \mathfrak{s}_{2,\bar{3}}+5\mathfrak{f}+\mathfrak{g},\\ \Pi\,.\,\mathfrak{G} &= \mathfrak{t}_{(23)}+6\mathfrak{f}+8\mathfrak{g} \quad = \mathfrak{t}_{2,3}+\mathfrak{f}+5\mathfrak{g} \quad = \mathfrak{t}_{2,\bar{3}}+4\mathfrak{f}+5\mathfrak{g}. \end{aligned}\right\} \tag{19·9}$$

Thus if we define $\quad \mathfrak{u} = \mathfrak{s}_{(23)}, \quad \mathfrak{v} = \mathfrak{t}_{(23)}, \quad \mathfrak{x} = \mathfrak{s}_{3,\bar{3}}, \quad \mathfrak{y} = \mathfrak{t}_{3,\bar{3}},$

we have from (19·9)

$$\left.\begin{aligned} \mathfrak{s}_{2,3} &= \mathfrak{u}+3\mathfrak{f}-2\mathfrak{g}, & \mathfrak{t}_{2,3} &= \mathfrak{v}+5\mathfrak{f}+3\mathfrak{g},\\ \mathfrak{s}_{2,\bar{3}} &= \mathfrak{u}+3\mathfrak{f}+\mathfrak{g}, & \mathfrak{t}_{2,\bar{3}} &= \mathfrak{v}+2\mathfrak{f}+3\mathfrak{g} \end{aligned}\right\} \tag{19·10}$$

and from the top line of (19·5) with (19·10)

$$\left.\begin{aligned} \Phi_{(2\bar{3})} &= \Phi_{(23)3}-3\mathfrak{u}+3\mathfrak{v},\\ \Phi_{2,3,\bar{3}} &= \Phi_{(23)3}+11\mathfrak{f}-\mathfrak{g}+2\mathfrak{u}+\mathfrak{v},\\ \Phi_{\bar{2},3} &= \Phi_{(23)3}+11\mathfrak{f}-\mathfrak{g}+2\mathfrak{u}+\mathfrak{v}+3\mathfrak{x}-\mathfrak{y}. \end{aligned}\right\} \tag{19·11}$$

But since the images in $\{f'\}$ of $\mathfrak{W}$, F, G, $\Phi_{\bar{2}}$ on $W_{3,2}$, are $\mathfrak{w}$, $\mathfrak{s}_{2,3}$, $\mathfrak{t}_{2,3}$, $\Phi_{\bar{2},3}$, we have from (17·4)

$$\Phi_{\bar{2},3} = 11\mathfrak{f}-\mathfrak{g}+2\mathfrak{u}+\mathfrak{v}+7\mathfrak{w},$$

and comparing this value with (19·11)

$$\Phi_{(23)3}+3\mathfrak{x} = 7\mathfrak{w}+\mathfrak{y}. \tag{19·12}$$

Now, to find a base for all dimensions on $W_{3,3}$, we can take, for each member of a base on $W_{3,3}$ (including itself and a point) first its image in $\{W\}$, then a variety tracing a line on each plane of the subsystem in $\{W\}$, for which clearly the image in $\{f'\}$ will serve, and finally one meeting the general plane of the subsystem is a point. As an example of a subvariety of $W_{3,3}$ meeting the general plane W in a point we shall take the linear system $|\mathfrak{Z}|$ of which $\Phi_{3,\bar{3}}$ is one, traced on Φ_3 by the primals co-residual to $\Phi_{\bar{3}}$. This, however, meets ∞^2 of the planes in lines $m_{\bar{2},3} = f'$; it is thus a birational image of $W_{3,2}$, but with the surface $\Phi_{\bar{2}}$ dilated; it is in fact easily seen (since the images on $\mathfrak{Z}$ of f, g on $W_{3,2}$ are $m_{2,3,\bar{3}}$, $n_{2,3,\bar{3}}$, of orders 2, 6) to be the projective model of the subsystem of $|12\Omega_1+2\Phi_2|$ on $W_{3,2}$ which has $\Phi_{\bar{2}}$ as simple base locus.

Table 4 gives a base of all dimensions on $W_{3,2}$ with the images of each element in $\{W\}$, in $\{f'\}$, and on $\mathfrak{Z}$; the only new notation is $|\mathfrak{O}|$, the net of fourfolds traced on Φ_3 by Ω_1, with characteristic system $\{\mathfrak{w}\}$.

It is thus clear that Ω_1, Φ_2, Φ_3 are a base for primals, $\mathfrak{W}$, $\mathfrak{Q}$, $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{Z}$, $\Phi_{(23)}$ for fourfolds, $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, $\mathfrak{x}$, $\Phi_{(23)3}$ for threefolds, W, F, G, R, I, J for surfaces, and f, g, h for curves.

All equivalences on $W_{3,2}$ are repeated in $\{\mathfrak{W}\}$, in $\{f'\}$, and on $\mathfrak{Z}$ (mod $\Phi_{\tilde{2},3}$). Intersections are repeated in $\{W\}$ as intersections on $W_{3,3}$, and in $\{f'\}$ as intersections on Φ_3 (or on $W_{3,3}$ of either member with something tracing the other on Φ_3). For the prime sections, since the residual section by a prime through Φ_3 is compounded with $\{W\}$ and cuts $\mathfrak{f}$, $\mathfrak{g}$ in $4W$, W respectively we have from the base and intersection theory in $\{W\}$

$$\Pi = 13\Omega_1 + 4\Phi_2 + \Phi_3, \tag{19·13}$$

and cutting this equivalence by Ω_1,

$$\Pi . \Omega_1 = 13\mathfrak{W} + 4\mathfrak{F} + \mathfrak{Q}. \tag{19·14}$$

TABLE 4

	in $\{W\}$	in $\{f'\}$	on $\mathfrak{Z}$
$W_{3,2}$	$W_{3,3}$	Φ_3	$\mathfrak{Z}$
Ω_1	Ω_1	$\mathfrak{Q}$	$\mathfrak{s}_{3,\tilde{3}} = \mathfrak{x}$
Φ_2	Φ_2	$\Phi_{2,3} = \Phi_{(23)} + 5\mathfrak{F} - 2\mathfrak{G}$	$\Phi_{2,3,\tilde{3}} = \Phi_{(23)3} + 11\mathfrak{f} - \mathfrak{g} + 2\mathfrak{u} + \mathfrak{v}$
W	$\mathfrak{W}$	$\mathfrak{w}$	$S_{3,\tilde{3}} = G$
F	$\mathfrak{F}$	$\mathfrak{s}_{2,3} = 3\mathfrak{f} - 2\mathfrak{g} + \mathfrak{u}$	$M_{2,3,\tilde{3}} = 5W + F + R + I$
G	$\mathfrak{G}$	$\mathfrak{t}_{2,3} = 5\mathfrak{f} + 3\mathfrak{g} + \mathfrak{v}$	$N_{2,3,\tilde{3}} = 21W + 8F + 4G + 3R + 2I + J$
f	$\mathfrak{f}$	$S_{2,3} = F$	$m_{2,3,\tilde{3}} = f + g$
g	$\mathfrak{g}$	$T_{2,3} = 5W + R$	$n_{2,3,\tilde{3}} = 5f + h$
point	W	f	point

Similarly, as a prime through $\mathfrak{Z}$ cuts F, $T_{2,3}$ residually in f, $4f$,

$$\Pi . \Phi_3 = 7\mathfrak{Q} + \mathfrak{Z} + \Phi_{2,3} = 5\mathfrak{F} - 2\mathfrak{G} + 7\mathfrak{Q} + \mathfrak{Z} + \Phi_{(23)} \tag{19·15}$$

and again cutting by Ω_1,

$$\Pi . \mathfrak{Q} = 7\mathfrak{w} + \mathfrak{s}_{3,\tilde{3}} + \mathfrak{s}_{2,3} = 3\mathfrak{f} - 2\mathfrak{g} + \mathfrak{u} + 7\mathfrak{w} + \mathfrak{x}. \tag{19·16}$$

We are now in a position to construct the complete intersection table (Table 5) for $W_{3,3}$. The prime section of each element of the base either is given above, or is trivial, or is found from (19·5) with the appropriate values of i, j, i', j'. The section of each element by Ω_1 is easily found; that by Φ_2 is clear except when the element lies on Φ_2, when it can be found as the section, on Φ_2, by the latter's characteristic system $|-3\mathfrak{F}+\mathfrak{G}|$. The section by Φ_3, where it is not obvious, is found from the linear identity (19·13) between the four columns. The sections by fourfolds and threefolds are obtained from those by primals, using the associativity of intersection, and expressing each fourfold or threefold as a single intersection, e.g.

$$\mathfrak{G} = (3\Omega_1 + \Phi_2) . \Phi_2, \quad \mathfrak{v} = (\Omega_1 + 2\Phi_2 + \Phi_3) . G$$

and so forth.

All the Φ loci except $\Phi_{\tilde{3}}$ have by now been expressed in terms of the base; and as $\Phi_3 . \Phi_{\tilde{3}} = \mathfrak{Z}$ we have

$$\Phi_{\tilde{3}} = 6\Omega_1 + 3\Phi_2 + \Phi_3. \tag{19·17}$$

TABLE 5

$W_{3,3}$	order 23,734	Ω_1 section Ω_1	Φ_2 section Φ_2	Φ_3 section Φ_3	prime section $13\Omega_1+4\Phi_2+\Phi_3$
Ω_1	1,138	$\mathfrak{W}$	$\mathfrak{F}$	$\mathfrak{Q}$	$13\mathfrak{W}+4\mathfrak{F}+\mathfrak{Q}$
Φ_2	1,490	$\mathfrak{F}$	$-3\mathfrak{F}+\mathfrak{G}$	$5\mathfrak{F}-2\mathfrak{G}+\Phi_{(23)}$	$6\mathfrak{F}+2\mathfrak{G}+\Phi_{(23)}$
Φ_3	2,980	$\mathfrak{Q}$	$5\mathfrak{F}-2\mathfrak{G}+\Phi_{(23)}$	$-15\mathfrak{F}+6\mathfrak{G}-6\mathfrak{Q}+\mathfrak{Z}-3\Phi_{(23)}$	$5\mathfrak{F}-2\mathfrak{G}+7\mathfrak{Q}+\mathfrak{Z}+\Phi_{(23)}$
$\mathfrak{W}$	34	0	$\mathfrak{f}$	$\mathfrak{w}$	$4\mathfrak{f}+\mathfrak{w}$
$\mathfrak{F}$	132	$\mathfrak{f}$	$-2\mathfrak{f}+\mathfrak{g}$	$3\mathfrak{f}-2\mathfrak{g}+\mathfrak{u}$	$8\mathfrak{f}+2\mathfrak{g}+\mathfrak{u}$
$\mathfrak{G}$	184	$\mathfrak{f}+\mathfrak{g}$	$-3\mathfrak{f}-2\mathfrak{g}$	$5\mathfrak{f}+3\mathfrak{g}+\mathfrak{v}$	$6\mathfrak{f}+8\mathfrak{g}+\mathfrak{v}$
$\mathfrak{Q}$	168	$\mathfrak{w}$	$3\mathfrak{f}-2\mathfrak{g}+\mathfrak{u}$	$-9\mathfrak{f}+6\mathfrak{g}-3\mathfrak{u}-6\mathfrak{w}+\mathfrak{x}$	$3\mathfrak{f}-2\mathfrak{g}+\mathfrak{u}+7\mathfrak{w}+\mathfrak{x}$
$\mathfrak{Z}$	1,182	$\mathfrak{x}$	$11\mathfrak{f}-\mathfrak{g}+2\mathfrak{u}+\mathfrak{v}+\Phi_{(23)3}$	$-33\mathfrak{f}+3\mathfrak{g}-6\mathfrak{u}-3\mathfrak{v}-7\mathfrak{w}-\mathfrak{x}-2\Phi_{(23)3}$	$11\mathfrak{f}-\mathfrak{g}+2\mathfrak{u}+\mathfrak{v}-7\mathfrak{w}+12\mathfrak{x}+2\Phi_{(23)3}$
$\Phi_{(23)}$	330	$\mathfrak{u}$	$-3\mathfrak{u}+\mathfrak{v}$	$\Phi_{(23)3}$	$\mathfrak{u}+4\mathfrak{v}+\Phi_{(23)3}$
$\mathfrak{f}$	7	0	W	F	$4W+F$
$\mathfrak{g}$	11	W	$-3W$	$5W+R$	$6W+R$
$\mathfrak{u}$	54	$2W+F$	$-4W-2F+R$	I	$10W+5F+4R+I$
$\mathfrak{v}$	54	$2W+F+R$	$-6W-3F-2R$	$6W+3F+J$	$8W+4F+5R+J$
$\mathfrak{w}$	6	0	F	$-3F+G$	$F+G$
$\mathfrak{x}$	73	G	$5W+3F+R+I$	$-15W-9F-G-3R-2I$	$5W+3F+12G+R+2I$
$\Phi_{(23)3}$	60	I	$6W+3F-3I+J$	$-18W-9F+3I-3J$	$6W+3F+4I+J$
W	1	0	0	f	f
F	3	0	f	$-2f+g$	$2f+g$
G	3	0	$f+g$	$-3f-2g$	$f+2g$
R	5	f	$-3f$	$3f+h$	$4f+h$
I	9	g	$3f-2g+h$	$-9f-3h$	$3f+5g+h$
J	9	$3f+g+h$	$-9f-3g-2h$	0	$3f+g+5h$
f	1	0	0	1	1
g	1	0	1	-3	1
h	1	1	-3	0	1

TABLE 5 (*cont.*)

	$\mathfrak{W}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{Q}$	$\mathfrak{Z}$	$\Phi_{(23)}$
$\mathfrak{W}$	0	0	W	0	G	$2W+F$
$\mathfrak{F}$	0	W	$-2W$	F	$5W+3F+R+I$	$-4W-2F+R$
$\mathfrak{G}$	W	$-2W$	$-3W$	$5W+F+R$	$21W+5F+3R+J$	$-6W-3F-2R$
$\mathfrak{Q}$	0	F	$5W+F+R$	$-3F+G$	$-15W-9F-G-3R-2I$	I
$\mathfrak{Z}$	G	$5W+3F+R+I$	$21W+5F+3R+J$	$-15W-9F-G-3R-2I$	$-93W-33F-12G-15R-7I-3J$	0
$\Phi_{(23)}$	$2W+F$	$-4W-2F+R$	$-6W-3F-2R$	I	0	$14W+7F-9R-3I+J$
$\mathfrak{f}$	0	0	0	0	$f+g$	f
$\mathfrak{g}$	0	0	0	f	$5f+h$	$-3f$
$\mathfrak{u}$	0	f	$-2f$	g	0	$-6f-2g+h$
$\mathfrak{v}$	f	$-2f$	$-3f$	$3f+g+h$	0	$-2f-3g-2h$
$\mathfrak{w}$	0	0	f	0	$-3f-2g$	g
$\mathfrak{x}$	0	$f+g$	$6f+g+h$	$-3f-2g$	$-24f-10g-3h$	0
$\Phi_{(23)3}$	g	$3f-2g+h$	$-6f-3g-2h$	$-9f-3h$	0	$7g$
W	0	0	0	0	1	0
F	0	0	0	0	-2	1
G	0	0	1	0	-3	0
R	0	0	0	1	0	-3
I	0	1	-2	-3	0	0
J	1	-2	-3	0	0	4

	$\mathfrak{f}$	$\mathfrak{g}$	$\mathfrak{u}$	$\mathfrak{v}$	$\mathfrak{w}$	$\mathfrak{x}$	$\Phi_{(23)3}$
$\mathfrak{f}$	0	0	0	0	0	0	1
$\mathfrak{g}$	0	0	0	0	0	1	-3
$\mathfrak{u}$	0	0	1	-2	0	0	0
$\mathfrak{v}$	0	0	-2	-3	1	0	7
$\mathfrak{w}$	0	0	0	1	0	0	-3
$\mathfrak{x}$	0	1	0	0	0	-3	0
$\Phi_{(23)3}$	1	-3	0	7	-3	0	-21

$$\Pi = 13\Omega_1+4\Phi_2+\Phi_3,$$
$$\Pi^2 = 169\mathfrak{W}+81\mathfrak{F}+6\mathfrak{G}+20\mathfrak{Q}+\mathfrak{Z}+5\Phi_{(23)},$$
$$\Pi^3 = 1431\mathfrak{f}+169\mathfrak{g}+108\mathfrak{u}+27\mathfrak{v}+302\mathfrak{w}+32\mathfrak{x}+7\Phi_{(23)3},$$
$$\Pi^4 = 8236W+2498F+686G+768R+200I+34J,$$
$$\Pi^5 = 17692f+4094g+1138h.$$

For the remaining condition loci, applying (17·3, 4) to $\{W\}$ we have

$$\Sigma_2 = 3\Omega_1 + \Phi_2, \quad \Omega_2 = 2\mathfrak{W} + \mathfrak{F}, \tag{19·18}$$

whence

$$\left.\begin{aligned} \Sigma_3 &= \Phi_{\bar{3}}.\Sigma_2 = 9\mathfrak{W} + 17\mathfrak{F} - \mathfrak{G} + 3\mathfrak{Q} + \Phi_{(23)},\\ \Omega_3 &= \Phi_{\bar{3}}.\Omega_2 - \mathfrak{a} = 6\mathfrak{f} + \mathfrak{u} + 2\mathfrak{w}. \end{aligned}\right\} \tag{19·19}$$

20. Line systems on $W_{3,n}$

We cannot of course fail to notice the analogy between the three line systems on $W_{3,3}$ and those on $W_{2,3}$, and that the images on $\overline{\Omega}_3$ of l_1, l_2, l_3 on $W_{2,3}$ are h, g, f, respectively; and we naturally expect to find on $W_{3,n}$ a sequence of n line systems of increasing dimensions, analogous to that on $W_{2,n}$.

We have on $W_{3,n}$ a subvariety $\mathfrak{w}^{(1,j)}$, where $j = \frac{1}{2}(3^{n-1}-1)$, given by the monomials $\mathbf{p}^{\alpha}\mathbf{D}_{\beta_1}\ldots\mathbf{D}_{\beta_j}$ in its parametrization, the other co-ordinates vanishing. Each generating line of this is clearly the unique line l_1 on the $\overline{\Omega}_n$ given by the same ratios $\mathbf{D}_x:\mathbf{D}_y:\mathbf{D}_z$; and thus this $\mathfrak{w}^{(1,j)}$ is the locus $\Phi_{(\overline{23\ldots n})}$ of images of sequences in which $\mathsf{P}_1\ldots\mathsf{P}_n$ are all proximate to P_0 and all collinear in the neighbourhood of P_0. This ∞^2 line system on $W_{3,n}$ we shall denote by $\{l_1\}$.

Just as on $W_{2,n}$, we now define the ∞^{2i} line system $\{l_i\}$ on $W_{3,n}$ to be the union of the line systems, images on all the $\infty^{2(i-1)}$ $W_{3,n-i+1}(\mathsf{P}_{i-1})$'s of the system $\{l_1\}$ on $W_{3,n-i+1}$. This system generates the locus $\Phi_{(i+1,\overline{i+2\ldots n})}$ of images of sequences $\mathsf{P}_0\ldots\mathsf{P}_n$ in which $\mathsf{P}_i, \mathsf{P}_{i+1}, \ldots, \mathsf{P}_n$ are all proximate to P_{i-1}, and all collinear in the neighbourhood of P_{i-1}. It is also clear that the images on each Ω_n of the lines $\{l_i\}$ on $W_{2,n}$ are a subsystem of $\{l_i\}$ on $W_{3,n}$. In particular the line system $\{l_n\}$ consists of all lines in all the $\infty^{2(n-1)}$ planes $\{W_{3,1}(\mathsf{P}_{n-1})\}$.

Obviously the line systems $\{l_i\}$ on all $W_{2,n}$'s $(n \geq i)$ are in one-one correspondence with each other, being mapped on the points of $W_{2,i-1}$. On $W_{3,n}$ the situation is more complicated; but it is still true that the line system $\{l_i\}$ on all $W_{3,n}$'s $(n \geq i)$ are in one-one correspondence with each other, as can be seen from the following argument.

Suppose any consecutive sequence $\mathsf{P}_0\ldots\mathsf{P}_{i-1}$ to be dilated, giving a variety $S_3^{(i)}$ on which the neighbourhood of P_{i-1} appears as a plane $W^{(i)}$, and in this plane consider any line $l^{(i)}$. Then for any $n \geq i$ we can define ∞^1 sequences $\mathsf{P}_0\ldots\mathsf{P}_n$ containing the subsequence $\mathsf{P}_0\ldots\mathsf{P}_{i-1}$ by taking as the explicit image $\mathsf{P}_i^{(i)}$ of P_i any point of $l^{(i)}$, and the implicit images $\mathsf{P}_{i+1}^{(i)}, \ldots, \mathsf{P}_n^{(i)}$ of the remaining points $\mathsf{P}_{i+1}, \ldots, \mathsf{P}_n$ (if any) consecutive to $\mathsf{P}_i^{(i)}$ along $l^{(i)}$; and these ∞^1 sequences are mapped by the points of a line l_i on $W_{3,n}$. The construction is equivalent to taking an arbitrary surface through $\mathsf{P}_0\ldots\mathsf{P}_{i-1}$, simple at P_{i-1}, and taking the rest of the sequence $\mathsf{P}_i\ldots\mathsf{P}_n$ on this surface and all proximate to P_{i-1}. The lines $\{l_i\}$ on $W_{3,n}$ are thus in one-one correspondence with what Semple calls the surface elements of the ith order at P_0.

A minimum order model for the ∞^2 line system $\{l_1\}$ is of course a plane; one for $\{l_i\}$ is a fibre space of planes over $W_{3,i-1}$, but this of course does not suffice to identify it with $W_{3,i}$; we shall see in fact that already for $i = 2$ the two fourfolds are quite different.

Let us define the fourfold $\mathfrak{S}_j^i$ as follows: We take $\mathfrak{w}^* = \mathfrak{w}^{(i,1)}$, on which we define as usual the lines $\{f^*\}$ and i-ic curves $\{g^*\}$, and the surfaces $|F^*|$ (ruled in $\{f^*\}$) and $|G^*|$ (generated by a pencil in $\{g^*\}$). We take also a surface C^*, mapped on a plane by all j-ic curves, the lines of the plane appearing as a homaloidal net $|c^*|$ of j-ic curves of C^*. The lines $\{f^*\}$ being mapped in the obvious way on the points of C^*, we join each corresponding point and line by a plane W^*, and those ∞^2 planes generate the fourfold $\mathfrak{S}^i_j$. Thus $W_{3,2}$ is $\mathfrak{S}^1_4$, and the locus $\Phi_{\bar{2}}$ on $W_{3,3}$ is $\mathfrak{S}^9_{13}$. We define the threefold $\mathfrak{q}^*$, generated by the ∞^1 planes W^* tracing a ruled surface F^* on w^* and a curve c^* on C^*, and the ruled surface R^* generated by lines (one in each plane of q^*) joining corresponding points of c^*, g^*; and we note that as any line in a plane W^* is equivalent to f^*, on R^*

$$g^*-c^* = (i-j)\,f^*. \tag{20·1}$$

The prime sections of $\mathfrak{S}^i_j$ are the system $|j\mathfrak{q}^*+\mathfrak{w}^*|$, and we easily construct the intersection table (Table 6), from which follows the equivalence

$$C^* = [(i-j)^2+i-j+1]\,W^*-(i-j+1)\,F^*+G. \tag{20·2}$$

Table 6

	$\mathfrak{q}^*$	$\mathfrak{w}^*$	$j\mathfrak{q}^*+\mathfrak{w}^*$
$\mathfrak{q}^*$	W^*	F^*	jW^*+F^*
$\mathfrak{w}^*$	F^*	$(i-j)\,F^*+G^*$	iF^*+G^*
W^*	0	f^*	f^*
F^*	f^*	$(i-j+1)\,f^*+g^*$	$(i+1)\,f^*+g^*$
G^*	f^*+g^*	$(i-j)\,f^*+(i-j+1)\,g^*$	$if^*+(i+1)\,g^*$
C^*	$(j-i)\,f+g^*$	0	$j(j-1)\,f^*+jg^*$
f^*	0	1	1
g^*	1	$i-j$	i

We see that the invariant intersections and equivalences depend only on the difference $i-j$, not on i, j separately; and we expect all $\mathfrak{S}^i_j$ for which this difference is the same to be equivalent; in fact it is easily seen that the projective model of the linear system

$$|(j+k)\,\mathfrak{q}^*+\mathfrak{w}^*| \quad \text{on} \quad \mathfrak{S}^i_j \quad \text{is} \quad \mathfrak{S}^{i+k}_{j+k}.$$

Thus a minimum order model is $\mathfrak{S}^1_{j-i+1}$ or $\mathfrak{S}^{i-j+1}_1$ according as $i \geqslant j$ or $i \leqslant j$. A striking difference between the two cases is that in the latter $\mathfrak{w}^*$ is unique while C^* varies in a linear system, in the former $\mathfrak{w}^*$ varies in a linear system and C^* is unique.

We now assert that the minimum order model of the ∞^4 line system $\{l_2\}$ on any $W_{3,n}$ $(n \geqslant 2)$ is S^3_1. We can prove this most simply for the system $\{g\}$ on $W_{3,3}$. Generated in $\{g\}$ we have the fourfolds $\mathfrak{Q}$, $\mathfrak{Z}$, the threefolds $\mathfrak{w}$, $\mathfrak{x}$, $\mathfrak{y}$, $\Phi_{(23)3}$, satisfying (19·12), and the ruled surfaces $G, I, T_{3,\bar{3}} = 2G-I$. The ∞^2 generators of $\Phi_{(23)3}$ and likewise those of each $\mathfrak{w}$ are mapped on a plane, while those of $\mathfrak{Z}$ are mapped on any $\mathfrak{w}^*$, the subsystems generating $\mathfrak{x}$, $\mathfrak{y}$, G, $T_{3,\bar{3}}$ being mapped by F^*, G^*, f^*, g^*; and each $\mathfrak{w}$ has a surface G in common with Z and a single generating line in common with $\Phi_{(23)3}$. We therefore identify the generating line system of $\Phi_{(23)3}$ with C^* (so that the curve c^* corresponds to a ruled surface I) and that of $\mathfrak{w}$ with W; and the relations

$$T_{3,\bar{3}}-I = 2G, \quad \Phi_{(23)3} = 7\mathfrak{w}-3\mathfrak{x}+\mathfrak{y}$$

lead to

$$g^*-c^* = 2f^*, \quad c^* = 7W^*-3F^*+G^*. \tag{20·3}$$

which compared with (20·1, 2) shows that $i-j=2$, and that the minimum order model is $\mathfrak{S}_1^3$.

The mapping on $\mathfrak{S}_1^3$ of the line system $\{f\}$ on $W_{3,2}$ is less immediately obvious, as the system generates $W_{3,2}$ multiply, and there is not the same simple correspondence between subsystems and the subvarieties they generate But it is easy to identify W^* with the ∞^2 lines in a plane W, and C^* with those on Φ_2, the latter having just one line in common with each W. For $\mathfrak{w}^*$, we want a subsystem containing a pencil in each W, and having no line in common with Φ_2; this is evidently provided by the lines meeting $\Phi_{\bar{2}}$, which is unisecant to W and does not meet Φ_2. The curves f^*, g^*, c^* thus represent a pencil of lines in W and the generators of the ruled surfaces $\bar{\Omega}_2$, F; and the relation $\Omega_2 = 2W+F$ gives us the first of (20·3), which is sufficient to establish that $i+j=2$. The other equation is (20·3) would correspond to

$$\Phi_2 = 7W - 3\Omega_1 + \Sigma_2,$$

but as W is of lower dimensions than the rest it drops out of the equivalence, which thus reduces to the second of (17·3).

We may note that an exactly similar procedure enables us to map the ∞^4 system of lines in generating planes of any $\mathfrak{S}_j^i$ by $\mathfrak{S}_{j'}^{i'}$, where

$$(i-j)+(i'-j') = -1.$$

Thus not only is $\mathfrak{S}_1^3$ the minimum order model for the line system $\{f\}$ on $W_{3,2}$, but $W_{3,2}$ is the minimum order model for the line system $\{f\}$ on $\mathfrak{S}_1^3$. The same sort of reciprocity continues to hold between $W_{3,n}$ and the minimum order model Ξ_n of the line system $\{l_n\}$, which like $W_{3,n}$ is a fibre space of planes over $W_{3,n-1}$; for on $W_{3,n+1}$ the line system $\{l_n\}$ consists of the generators g of all the $\mathfrak{w}^{(1,1)}$'s, images on $W_{3,2}(\mathsf{P}_{n-2})$ of Φ_2 on $W_{3,2}$; the lines in each $\mathfrak{w}^{(1,1)}$ are mapped by a plane of Ξ_n, and the generators of a ruled cubic G on $\mathfrak{w}^{(1,1)}$ by a line in this plane. But every G has its directrix line in a unique plane $W_{3,1}(\mathsf{P}_n)$; the aggregate of ruled cubics G on $W_{3,n+1}$, i.e. of lines on Ξ_n, is thus in one-one correspondence with the points of $W_{3,n}$.

PART III. $W^*_{r,n}$; IN PARTICULAR $W^*_{2,n}$

21. Homogeneous co-ordinates in the plane

For the study of $W^*_{2,n}$, the aggregate of plane sequences $\mathsf{P}_0 \dots \mathsf{P}_n$, not only with a fixed origin P_0, but with P_0 anywhere in the plane, in order to obtain results in a form that has some sort of invariance with respect to collineations in the plane, it is natural to use a homogeneous co-ordinate system, which we shall denote by (u^x, u^y, u^z), and to specify the generic point of an algebroid branch in the form

$$u^\alpha = u_0^\alpha + u_1^\alpha t + u_2^\alpha t^2 + u_3^\alpha t^3 + \dots, \tag{21·1}$$

where (u_0^x, u_0^y, u_0^z) are not all zero, and are the co-ordinates of the origin of the branch. For the generic branch, we can form an affine co-ordinate system (x, y) with origin at the origin of the branch by defining

$$x = \frac{u^x}{u^z} - \frac{u_0^x}{u_0^z}, \quad y = \frac{u^y}{u^z} - \frac{u_0^y}{u_0^z}; \tag{21·2}$$

and by straightforward division in (21·1) we obtain the generic point of the same branch in a form in which the coefficient a_i, b_i of (3·1) are explicit functions of the coefficients in (21·1). In fact, defining the determinants

$$q_{\alpha(ij)} = \epsilon_{\alpha\beta\gamma} u_i^\beta u_j^\gamma$$

it is easily seen that

$$\left.\begin{aligned}
a_1 &= q_{y(01)}/(u_0^z)^2,\\
a_2 &= (-u_1^z q_{y(01)} + u_0^z q_{y(02)})/(u_0^z)^3,\\
a_3 &= \{[(u_1^z)^2 - u_0^z u_2^z]\, q_{y(01)} - u_0^z u_1^z q_{y(02)} + (u_0^z)^2 q_{y(03)}\}/(u_0^z)^4,\\
a_4 &= \{-[(u_0^z)^2 u_3^z - 2u_0^z u_1^z u_2^z + (u_1^z)^3]\, q_{y(01)}\\
&\quad + u_0^z[(u_1^z)^2 - u_0^z u_2^z]\, q_{y(02)} - (u_0^z)^2 u_1^z q_{y(03)} + (u_0^z)^3 q_{y(04)}\}/(u_0^z)^5,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \qquad (21\cdot3)$$

and $b_1, b_2, \ldots$ are similarly expressed with $-q_{x(0i)}$ in place of $q_{y(0i)}$. Further, defining the cubic determinants

$$D^*_{ijk} = \epsilon_{\alpha\beta\gamma} u_i^\alpha u_j^\beta u_k^\gamma,$$

we find for the determinants D_{ij} of § 3 the values

$$\left.\begin{aligned}
D_{12} &= D^*_{012}/(u_0^z)^3,\\
D_{13} &= (-u_1^z D^*_{012} + u_0^z D^*_{013})/(u_0^z)^4,\\
D_{14} &= \{[(u_1^z)^2 - u_0^z u_2^z]\, D^*_{012} - u_0^z u_1^z D^*_{013} + (u_0^z)^2 D^*_{014}\}/(u_0^z)^5,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \qquad (21\cdot4)$$

Substituting the values of $a_1, a_2, \ldots$ from (21·3) and those of $D_{12}, D_{13}, \ldots$ from (21·4) in any t-invariant (a, D) form of the branch (3·1), we obtain a fraction whose denominator is a power of u_0^z, and whose numerator is a t-invariant, of the same weight and rank, of the branch (21·1), and is a form in $(u_0^z, u_1^z, \ldots)$, in $(q_{y(01)}, q_{y(02)}, \ldots)$, and in $(D^*_{012}, D^*_{013}, \ldots)$, which we may call a (u^z, q_y, D^*) form. Since, however, (u_i^x, u_i^y, u_i^z) are the components of a contravariant tensor, with respect to a general linear transformation of the co-ordinates, $(q_{x(ij)}, q_{y(ij)}, q_{z(ij)})$ those of a covariant tensor, and D^*_{ijk} a scalar, our (u^z, q_y, D^*) form is one component of a t-invariant tensor, namely that in which all the covariant indices are y and all the contravariant indices z; and the similar expressions for the tensor companions of the original (a, D) form are certain other components of the same tensor, namely, those in which some or all of the covariant indices are x instead of y, the contravariant indices being still all z.

Thus we easily calculate

$$\left.\begin{aligned}
&\mathbf{D} = \frac{\mathbf{D}^*}{(u_0^z)^3};\\
&\mathbf{G} = \frac{\mathbf{G}^{*z}_y}{(u_0^z)^6}, \quad \mathbf{G}_1 = -\frac{\mathbf{G}^{*z}_x}{(u_0^z)^6};\\
&\mathbf{I} = \frac{\mathbf{I}^{*zz}_{yy}}{(u_0^z)^9}, \quad \mathbf{I}_1 = -2\frac{\mathbf{I}^{*zz}_{xy}}{(u_0^z)^9}, \quad \mathbf{I}_2 = \frac{\mathbf{I}^{*zz}_{xx}}{(u_0^z)^9};\\
&\mathbf{J} = \frac{\mathbf{J}^{*zz}_{yy}}{(u_0^z)^{12}}, \quad \mathbf{J}_1 = -2\frac{\mathbf{J}^{*zz}_{xy}}{(u_0^z)^{12}}, \quad \mathbf{J}_2 = \frac{\mathbf{J}^{*zz}_{xx}}{(u_0^z)^{12}};
\end{aligned}\right\} \qquad (21\cdot5)$$

where (abbreviating $q_{\alpha(0i)}$ for simplicity to $q_{\alpha i}$)

$$\left.\begin{aligned}
\mathbf{D} &= D^*_{012},\\
\mathbf{G}^{*\alpha}_{\beta} &= (u^\alpha_1 q_{\beta 1} - 2u^\alpha_0 q_{\beta 2})\, D^*_{012} + u^\alpha_0 q_{\beta 1} D^*_{013},\\
\mathbf{I}^{*\alpha\beta}_{\gamma\delta} &= \{\tfrac{1}{2}(u^\alpha_0 u^\beta_2 + 2u^\alpha_1 u^\beta_1 + u^\alpha_2 u^\beta_0)\, q_{\gamma 1} q_{\delta 1}\\
&\quad - \tfrac{5}{4}(u^\alpha_0 u^\beta_1 + u^\alpha_1 u^\beta_0)\,(q_{\gamma 1} q_{\delta 2} + q_{\gamma 2} q_{\delta 1}) - u^\alpha_0 u^\beta_0 (q_{\gamma 1} q_{\delta 3} - 5q_{\gamma 2} q_{\delta 2} + q_{\gamma 3} q_{\delta 1})\}\, D^*_{012}\\
&\quad + \{(u^\alpha_0 u^\beta_1 + u^\alpha_1 u^\beta_0)\, q_{\gamma 1} q_{\delta 1} - \tfrac{3}{2} u^\alpha_0 u^\beta_0 (q_{\gamma 1} q_{\delta 2} + q_{\gamma 2} q_{\delta 1})\}\, D^*_{013} + u^\alpha_0 u^\beta_0 q_{\gamma 1} q_{\delta 1} D^*_{014},\\
\mathbf{J}^{*\alpha\beta}_{\gamma\delta} &= \{\tfrac{1}{2}(u^\alpha_0 u^\beta_2 - 2u^\alpha_1 u^\beta_1 - u^\alpha_2 u^\beta_0)\, q_{\gamma 1} q_{\delta 1}\\
&\quad + \tfrac{3}{4}(u^\alpha_0 u^\beta_1 + u^\alpha_1 u^\beta_0)\,(q_{\gamma 1} q_{\delta 2} + q_{\gamma 2} q_{\delta 1}) - u^\alpha_0 u^\beta_0 (q_{\gamma 1} q_{\delta 2} + 3q_{\gamma 2} q_{\delta 3} + q_{\gamma 3} q_{\delta 1})\}\, D^{*2}_{012}\\
&\quad - \{(u^\alpha_0 u^\beta_1 + u^\alpha_1 u^\beta_0)\, q_{\gamma 1} q_{\delta 1} - \tfrac{5}{2} u^\alpha_0 u^\beta_0 (q_{\gamma 1} q_{\delta 2} + q_{\gamma 2} q_{\delta 1})\}\, D_{012} D_{013}\\
&\quad - u^\alpha_0 u^\beta_0 q_{\gamma 1} q_{\delta 1} (D^*_{012} D^*_{014} - 2D^{*2}_{013}).
\end{aligned}\right\} \tag{21·6}$$

These satisfy of course a good many identities, of which (5·1, ..., 6) are only a few; (5·4, 5) for instance are included in

$$\mathbf{J}^{*\alpha\beta}_{\gamma\delta} = \mathbf{D}^* \mathbf{I}^{*\alpha\beta}_{\gamma\delta} - \mathbf{G}^{*\alpha}_{\gamma} \mathbf{G}^{*\beta}_{\delta} - \mathbf{G}^{*\alpha}_{\delta} \mathbf{G}^{*\beta}_{\gamma}. \tag{21·7}$$

It is obvious that $\mathbf{u}^\alpha = u^\alpha_0$, the co-ordinates of the origin P_0 of the branch, are t-invariants of rank 0 and weight 0, and that $\mathbf{q}_\beta = q_{\beta(01)}$, the co-ordinates of the tangent $\mathsf{P}_0\mathsf{P}_1$, are t-invariants of rank 1 and weight 1. All the further principal t-invariants, of rank $\geqslant 2$, are the components of tensors with the same number of contravariant and covariant indices; we have seen that this is the case with $\mathbf{D}^*$, the only principal t-invariant of rank 2, which has no indices; with $\mathbf{G}^{*\alpha}_{\beta}$, the only one of rank 3, which has one of each; and with $\mathbf{I}^{*\alpha\beta}_{\gamma\delta}, \mathbf{J}^{*\alpha\beta}_{\gamma\delta}$, the only principal t-invariants of rank 4, which have two of each. We can in fact see that it is true generally; for if $\mathbf{F}$ is a t-invariant (a, D) form, of weight s, of degree h in $(a_1, \ldots)$ and of degree k in $(D_{12}, \ldots)$, the numerator $\mathbf{F}^*$ in the expression for $\mathbf{F}$ in terms of (21·1) will be a form of degree h in $(q_{y(01)}, \ldots)$ and of degree k in $(D_{012}, \ldots)$; also, as the sum of the suffixes i, j in all the factors a_i, D_{ij} in any term of $\mathbf{F}$ is equal to s, and as the numerators in the expressions (21·3, 4) for a_i, D_{1j} are forms in $(u^z_0, \ldots)$ of degrees $i-1, j-3$, respectively, $\mathbf{F}^*$ is a form in $(u^z_0, \ldots)$ of degree

$$s - h - 3k = h$$

by (5·7). The tensor of which $\mathbf{F}^*$ is one component thus has h contravariant as well as h covariant indices.

Further, it is worth remarking that if $\mathbf{F}$ is of rank n, since it is a form in $(a_1, \ldots, a_{n-1})$ and in $(D_{12}, \ldots, D_{1n})$, $\mathbf{F}^*$ is a form in $(u^z_0, \ldots, u^z_{n-2})$, in $(q_{y1}, \ldots, q_{y,n-1})$ and in $(D^*_{012}, \ldots, D^*_{01n})$.

The identities satisfied by the t-invariant tensors of rank $\leqslant 3$ will be required when we deal with the parametrization of $W^*_{2,3}$; they are the following

$$\left.\begin{aligned}
\mathbf{u}^\alpha \mathbf{q}_\alpha = \mathbf{u}^\alpha \mathbf{G}^{*\beta}_{\alpha} = \mathbf{q}_\alpha \mathbf{G}^{*\alpha}_{\beta} = \mathbf{G}^{*\alpha}_{\alpha} = 0,\\
\epsilon_{\alpha\gamma\delta} \mathbf{u}^\gamma \mathbf{G}^{*\delta}_{\beta} = \mathbf{q}_\alpha \mathbf{q}_\beta \mathbf{D}^*, \quad \epsilon^{\alpha\gamma\delta} \mathbf{q}_\gamma \mathbf{G}^{*\beta}_{\delta} = -2\mathbf{u}^\alpha \mathbf{u}^\beta \mathbf{D}^{*2},
\end{aligned}\right\} \tag{21·8}$$

of which the last (with $\alpha = \beta = z$) is (5·1).

The t-invariants of a singular branch can be similarly treated. Denoting the first $q_{\alpha(0i)}$ whose components are not all 0 (co-ordinates of the tangent $\mathsf{P}_0\mathsf{P}_1$) by $\mathbf{q}'_\alpha$, and the first D_{0ij}

which is not 0 by $\mathbf{D}'^*$, and similarly in the affine form the first pair a_i, b_i which are not both 0 by $\mathbf{a}'$, $\mathbf{b}'$, and the first D_{ij} which is not 0 by $\mathbf{D}'$, we see that in every case

$$\mathbf{a}' = \frac{\mathbf{q}'_y}{(\mathbf{u}^z)^2}, \quad \mathbf{b}' = -\frac{\mathbf{q}'_y}{(\mathbf{u}^z)^2}, \quad \mathbf{D}' = \frac{\mathbf{D}'^*}{(\mathbf{u}^z)^3}$$

and $\mathbf{D}'^*$ is the only t-invariant of rank 2; where for the

$$\left.\begin{array}{llll} \text{cuspidal branch, species 1:} & \mathbf{q}'_\alpha = q_{\alpha(02)}, & \mathbf{D}'^* = D^*_{023}; \\ \text{cuspidal branch, species 2:} & \mathbf{q}'_\alpha = q_{\alpha(02)}, & \mathbf{D}'^* = D^*_{024}; \\ \text{cubic branch, species 1:} & \mathbf{q}'_\alpha = q_{\alpha(03)}, & \mathbf{D}'^* = D^*_{034}; \\ \text{cubic branch, species 2:} & \mathbf{q}'_\alpha = q_{\alpha(03)}, & \mathbf{D}'^* = D^*_{035}. \end{array}\right\} \tag{21·9}$$

The t-invariant $\mathbf{G}' = -3a_3D_{23} + 2a_2D_{24}$ of the cuspidal branch of species 1, and its tensor companion $\mathbf{G}'_1$ (of rank 4) are given by

$$\mathbf{G}' = \mathbf{G}'^{*z}_y/(\mathbf{u}^z)^6, \quad \mathbf{G}'_1 = -\mathbf{G}'^{*z}_x/(\mathbf{u}^z)^6,$$

where $$\mathbf{G}'^{*\alpha}_\beta = (u^\alpha_1 q_{\beta 2} - 3u^\alpha_0 q_{\beta 3})\, D^*_{023} + 2u^\alpha_0 q_{\beta 2} D^*_{024}. \tag{21·10}$$

Similarly the t-invariant $\mathbf{G}'' = -2a_3D_{24} + a_2D_{25}$ of the cuspidal branch of species 2 and its tensor companion $\mathbf{G}''_1$ (of rank 3) are given by

$$\mathbf{G}'' = \mathbf{G}''^{*z}_y/(\mathbf{u}^z)^6, \quad \mathbf{G}''_1 = -\mathbf{G}''^{*z}_x/(\mathbf{u}^z)^6,$$

where $$\mathbf{G}''^{*\alpha}_\beta = (u^\alpha_1 q_{\beta 2} - 2u^\alpha_0 q_{\beta 3})\, D_{024} + u^\alpha_0 q_{\beta 2} D^*_{025}. \tag{21·11}$$

We note also that $$\epsilon_{\alpha\gamma\delta}\mathbf{u}^\gamma \mathbf{G}''^{*\delta}_\beta = \epsilon^{\alpha\gamma\delta}\mathbf{q}_\gamma \mathbf{G}''^{\beta *}_\delta = 0,$$

so that we can write $$\mathbf{G}''^{*\alpha}_\beta = \mathbf{u}^\alpha \mathbf{q}_\beta \mathbf{D}''^*, \tag{21·12}$$

where $\mathbf{D}''^*$ is not a polynomial in the coefficients in (21·1), but a t-invariant rational function of them, like $\mathbf{D}''_\beta$ in § 15, to which it plays a very comparable role.

Of course, the homogeneous parametrization (21·1) still represents the same branch if the three series are all multiplied by any one series, with non-vanishing constant term c say; and the power series for x, y obtained by substituting from (21·1) in (21·2) are unchanged. Any homogeneous t-invariant $\mathbf{F}^*$, of degree h in $(\mathbf{u}^\alpha_0, \ldots, u^\alpha_{n-2})$ and in $(q_{\alpha 1}, \ldots, q_{\alpha n-1})$ and of degree k in $(D_{012}, \ldots, D_{01n})$ is therefore unchanged by this multiplication, except that it is obviously multiplied by c^m, where $m = 3(h+k)$; and the expression for the corresponding affine t-invariant $\mathbf{F}$ is

$$\mathbf{F} = \mathbf{F}^*/(\mathbf{u}^z)^m,$$

since the right-hand member must be homogeneous of degree zero. This applies as well to the t-invariants of singular as of non-singular branches. We shall call m the homogeneous degree of $\mathbf{F}^*$ (or $\mathbf{F}$); it satisfies, if s denotes the weight (except of course for $\mathbf{u}^*$)

$$s \leqslant m \leqslant 2s$$

with $m = 2s$ only in the case of $\mathbf{q}_\alpha$, $m = s$ only in that of $\mathbf{D}^*$.

22. Parametrization of $W_{2,n}$

We can now obtain the parametrization of $W^*_{2,n}$ from that of $W_{2,n}$ by the following procedure. We first substitute for each of the t-invariant monomials in the parametrization of $W_{2,n}$ its value in terms of the coefficients in (21·1), as found in the last section. Each monomial then becomes a fraction, whose numerator is a monomial in the homogeneous t-invariants just found, and whose denominator is a power of $\mathbf{u}^z$; and the highest power is $(\mathbf{u}^z)^{3^n-1}$, in the monomials arising from those in $(\mathbf{a}, \mathbf{b})$ only, since all the monomials have the same weight $\frac{1}{2}(3^n-1)$, and only these have homogeneous degree as high as twice their weight. Thus, multiplying throughout by $(\mathbf{u}^z)^{3^n}$, we obtain a set of monomials in the homogeneous t-invariants, in which the lowest exponent of $\mathbf{u}^z$ is 1. These are of course certain components of a well-defined set of t-invariant tensors, namely, those components in which all the contravariant indices are z and all the covariant indices x or y; and adjoining all the remaining components of these tensors we obtain an enlarged set of monomials which provide the parametrization of $W^*_{2,n}$.

Thus for $W^*_{2,1}$, from the monomials $\mathbf{a}$, $\mathbf{b}$ which parametrize the line $W_{2,1}$ we obtain by direct substitution the fractions $\mathbf{q}_y/(\mathbf{u}^z)^2$, $-\mathbf{q}_x/(\mathbf{u}^z)^2$; multiplying by $(\mathbf{u}^z)^3$ we have $\mathbf{u}^z\mathbf{q}_y$, $-\mathbf{u}^z\mathbf{q}_x$; and adjoining the remaining components of the tensor $\mathbf{u}^\alpha\mathbf{q}_\beta$ we have the familiar parametrization of $\mathfrak{w}^{(1,1)} = W^*_{2,1}$ in the form

$$X^\alpha_\beta = \mathbf{u}^\alpha\mathbf{q}_\beta \quad (X^\alpha_\alpha = 0). \tag{22·1}$$

Again, from the monomials

$$\mathbf{aD}, \quad \mathbf{bD}, \quad \mathbf{a}^4, \quad \mathbf{a}^3\mathbf{b}, \quad \mathbf{a}^2\mathbf{b}^2, \quad \mathbf{ab}^3, \quad \mathbf{b}^4$$

in the parametrization of $W_{2,2}$ we obtain by direct substitution

$$\frac{\mathbf{q}_y\mathbf{D}}{(\mathbf{u}^z)^5}, \quad \frac{\mathbf{q}_x\mathbf{D}}{(\mathbf{u}^z)^5}, \quad \frac{\mathbf{q}_y^4}{(\mathbf{u}^z)^8}, \quad \frac{\mathbf{q}_x\mathbf{q}_y^3}{(\mathbf{u}^z)^8}, \quad \frac{\mathbf{q}_x^2\mathbf{q}_y^2}{(\mathbf{u}^z)^8}, \quad \frac{\mathbf{q}_x^3\mathbf{q}_y}{(\mathbf{u}^z)^8}, \quad \frac{\mathbf{q}_x^4}{(\mathbf{u}^z)^8}$$

(apart from sign); and multiplying by $(\mathbf{u}^z)^9$ these become

$$(\mathbf{u}^z)^4\,\mathbf{q}_y\mathbf{D}, \quad (\mathbf{u}^z)^4\,\mathbf{q}_x\mathbf{D}, \quad \mathbf{u}^z\mathbf{q}_y^4, \quad \mathbf{u}^z\mathbf{q}_x\mathbf{q}_y^3, \quad \mathbf{u}^z\mathbf{q}_x^2\mathbf{q}_y^2, \quad \mathbf{u}^z\mathbf{q}_x^3\mathbf{q}_y, \quad \mathbf{u}^z\mathbf{q}_x^4.$$

Adjoining the remaining components of these tensors we have what is in effect Study's and Gherardelli's parametrization of $W_{2,2}$, namely

$$X^{\alpha\beta\gamma\delta}_\epsilon = \mathbf{u}^\alpha\mathbf{u}^\beta\mathbf{u}^\gamma\mathbf{u}^\delta\mathbf{q}_\epsilon\mathbf{D}^*, \quad Y^\epsilon_{\alpha\beta\gamma\delta} = \mathbf{u}^\epsilon\mathbf{q}_\alpha\mathbf{q}_\beta\mathbf{q}_\gamma\mathbf{q}_\delta, \tag{22·2}$$

satisfying of course the linear identities

$$X^{\alpha\beta\gamma\delta}_\delta = 0, \quad Y^\delta_{\alpha\beta\gamma\delta} = 0$$

in consequence of the first identity in (21·8). Treating in the same way the parametrization $(\mathbf{a}', \mathbf{b}', 0, 0, 0, 0, 0)$ of the sequences of type 2 (cuspidal branches) on $W_{2,2}$ we have

$$X^{\alpha\beta\gamma\delta}_\epsilon = \mathbf{u}'^\alpha\mathbf{u}'^\beta\mathbf{u}'^\gamma\mathbf{u}'^\delta\mathbf{q}'_\epsilon, \quad Y^\epsilon_{\alpha\beta\gamma\delta} = 0,$$

and putting $\mathbf{D}^* = 0$ for sequences of type $\bar{2}$ (inflected branches) we have

$$X^{\alpha\beta\gamma\delta}_\epsilon = 0, \quad Y^{\epsilon\beta\gamma\delta}_\alpha = \mathbf{u}^\epsilon\mathbf{q}_\alpha\mathbf{q}_\beta\mathbf{q}_\gamma\mathbf{q}_\delta$$

as the parametrizations of the loci of images on $W^*_{2,2}$ of sequences of these two special types, the only ones that arise for $n = 3$. We shall denote these loci on $W_{2,2}$ by Ψ_2, $\Psi_{\bar{2}}$; and we shall use Ψ with the corresponding suffixes to denote the locus of images on $W^*_{r,n}$ of sequences of any special type. Ψ_2 is a $\mathfrak{w}^{(4,1)}$, and $\Psi_{\bar{2}}$ a $\mathfrak{w}^{(1,4)}$ and $W_{2,2}$ is generated by lines joining corresponding points of these two threefolds. We note in passing that this is projectively indentical with the locus $\Phi_{(23)}$ on $W_{3,3}$, the threefolds Ψ_2 and $\Psi_{\bar{2}}$ corresponding to $\Phi_{(23)3}$ and $\Phi_{(2\bar{3})}$, respectively; and that this is precisely what we should expect, as a first dilation applied to the set of all sequences $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$ of type (23) (including the two more special cases of this) with origin P_0 gives us just the set of all sequences $\mathsf{P}_1^{(1)}\mathsf{P}_2^{(1)}\mathsf{P}_3^{(1)}$ in the plane mapping the first neighbourhood of P_0, and that if the original sequence is of type (23) 3 or $(2\overline{3})$, the dilated sequence will be of type 2 or $\overline{2}$, respectively.

Turning now to the parametrization of $W_{2,3}$ we start from the monomials

$$\mathbf{a}^{5-i}\mathbf{b}^i\mathbf{D}\mathbf{G}, \quad \mathbf{a}^{5-i}\mathbf{b}^i\mathbf{D}\mathbf{G}_1, \quad \mathbf{a}^{8-i}\mathbf{b}^i\mathbf{G}, \quad \mathbf{a}^{8-i}\mathbf{b}^i\mathbf{G}_1,$$

$$\mathbf{a}^{1-i}\mathbf{b}^i\mathbf{D}^4, \quad \mathbf{a}^{4-i}\mathbf{b}^i\mathbf{D}^3, \quad \mathbf{a}^{7-i}\mathbf{b}^i\mathbf{D}^2, \quad \mathbf{a}^{10-i}\mathbf{b}^i\mathbf{D}, \quad \mathbf{a}^{13-i}\mathbf{b}^i$$

which on direct substitution give the fractions (apart from sign)

$$\frac{\mathbf{q}_x^i\mathbf{q}_y^{5-i}\mathbf{D}^*\mathbf{G}_y^{*z}}{(\mathbf{u}^z)^{19}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{5-i}\mathbf{D}^*\mathbf{G}_x^{*z}}{(\mathbf{u}^z)^{19}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{8-i}\mathbf{G}_y^{*z}}{(\mathbf{u}^z)^{22}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{8-i}\mathbf{G}_x^{*z}}{(\mathbf{u}^z)^{22}},$$

$$\frac{\mathbf{q}_x^i\mathbf{q}_y^{1-i}\mathbf{D}^{*4}}{(\mathbf{u}^z)^{14}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{4-i}\mathbf{D}^{*3}}{(\mathbf{u}^z)^{17}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{7-i}\mathbf{D}^{*2}}{(\mathbf{u}^z)^{20}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{10-i}\mathbf{D}^*}{(\mathbf{u}^z)^{23}}, \quad \frac{\mathbf{q}_x^i\mathbf{q}_y^{13-i}}{(\mathbf{u}^z)^{26}};$$

multiplying all these by $(\mathbf{u}^z)^{27}$ and adjoining all the remaining components of each tensor, we obtain the following set of monomials, which we shall take for the parametrization of $W^*_{2,3}$

$$\left.\begin{aligned}
X^{\alpha_1\ldots\alpha_8,\,\gamma}_{\beta_1\ldots\beta_5,\,\delta} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_8}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_5}\mathbf{D}^*\mathbf{G}^{*\gamma}_{\delta},\\
X^{\alpha_1\ldots\alpha_5,\,\gamma}_{\beta_1\ldots\beta_8,\,\delta} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_5}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_8}\mathbf{G}^{*\gamma}_{\delta},\\
Y^{\alpha_1\ldots\alpha_{13}}_{\beta} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_{13}}\mathbf{q}_{\beta}\mathbf{D}^{*4},\\
Y^{\alpha_1\ldots\alpha_{10}}_{\beta_1\ldots\beta_4} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_{10}}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_4}\mathbf{D}^{*3},\\
Y^{\alpha_1\ldots\alpha_7}_{\beta_1\ldots\beta_7} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_7}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_7}\mathbf{D}^{*2},\\
Y^{\alpha_1\ldots\alpha_4}_{\beta_1\ldots\beta_{10}} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_4}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_{10}}\mathbf{D}^*,\\
Y^{\alpha}_{\beta_1\ldots\beta_{13}} &= \mathbf{u}^{\alpha}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_{13}}.
\end{aligned}\right\} \qquad (22\cdot3)$$

The proof that this does indeed furnish a parametrization of $W^*_{2,3}$ is precisely similar to that which would serve for the derivation of any $W^*_{2,n}$ from $W_{2,n}$, and is as follows.

In the first place, the co-ordinates (22·3) are all of the same weight, 13, and of the same homogeneous degree, 27. Thus every simple branch (21·1) has a unique image point in the space in which (22·3) are homogeneous co-ordinates; and this is the same for all branches through the same free sequence $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$, and can thus be regarded as an image point for the sequence.

Now the co-ordinates (22·3) satisfy a good many linear identities: in the first place, by the top line of (21·8), all the expressions obtained by contracting any covariant with any contravariant index in any one of the seven co-ordinate tensors vanish; there are also many alternating relations; as well as those that express the symmetry of each of these tensors in its contravariant α indices, and also in its covariant β indices, we have by the bottom line of (21·8)

$$\left.\begin{aligned}
\epsilon_{\alpha_8\gamma\kappa}X^{\alpha_1\ldots\alpha_8,\,\gamma}_{\beta_1\ldots\beta_5,\,\delta} &= Y^{\alpha_1\ldots\alpha_7}_{\beta_1\ldots\beta_5\,\delta\kappa}, \qquad \epsilon^{\beta_5\,\delta\kappa}X^{\alpha_1\ldots\alpha_8,\,\gamma}_{\beta_1\ldots\beta_5,\,\delta} = -2Y^{\alpha_1\ldots\alpha_8\,\gamma\kappa}_{\beta_1\ldots\beta_4},\\
\epsilon_{\alpha_5\,\gamma\kappa}X^{\alpha_1\ldots\alpha_5,\,\gamma}_{\beta_1\ldots\beta_8,\,\delta} &= Y^{\alpha_1\ldots\alpha_4}_{\beta_1\ldots\beta_8\,\delta\kappa}, \qquad \epsilon^{\beta_8\,\delta\kappa}X^{\alpha_1\ldots\alpha_5,\,\gamma}_{\beta_1\ldots\beta_8,\,\delta} = -2Y^{\alpha_1\ldots\alpha_5\,\gamma\kappa}_{\beta_1\ldots\beta_7}.
\end{aligned}\right\} \qquad (22\cdot4)$$

The image points of all sequences with origin at a generic point $(\xi, \eta, 1)$ satisfy a number of linear equations, analogous to (14·12), in which the coefficients are monomials in (ξ, η), expressing all co-ordinates of which any of the α indices are x or y in terms of those in which all the α indices are z. All the co-ordinates of the image point of a sequence with origin at $(\xi, \eta, 1)$ are in fact expressible linearly (with coefficients rational in (ξ, η)) in terms of those in which all the contravariant indices are z and all the covariant indices x or y; for when those with all combinations of α indices (but still with $\gamma = z$ for the X co-ordinates, and with all covariant indices x or y) have been found from these (ξ, η) equations, those with $\gamma = x, y$ and with the same covariant indices as before are given by the two right-hand equations (22·4), and finally those in which any covariant indices are z by the contraction identities.

This means of course that the images of sequences with a given origin P_0 all lie in a subspace of that in which (22·3) are the homogeneous co-ordinates, the equations of this subspace being just those (with coefficients involving the co-ordinates of P_0) that express the remaining co-ordinates (22·3) in terms of those in which all contravariant indices are z and all covariant indices x or y; these latter are accordingly a co-ordinate system in the subspace. Two such subspaces corresponding to different origins P_0 have no point in common, as all the co-ordinates involve $\mathbf{u}^{\alpha}$, and thus no image point of a free sequence with one origin coincides with any image point of a free sequence with any other origin. But for any given origin, the co-ordinates in which all contravariant indices are z and all covariant indices x or y are precisely those which are first obtained, as proportional to the monomials in the affine t-invariants that parametrize $W_{3,n}$; the locus of images obtained above of sequences with a given origin P_0 is thus no other than $W_{3,n}$; we shall call it $W_{3,n}(\mathsf{P}_0)$.

The algebraic variety of which (22·3) gives a generic point is thus generated by an ∞^2 congruence $\{W_{2,3}(\mathsf{P}_0)\}$; every point of the variety lies on one and only one member of the congruence, and is, as such, the image of a sequence with origin at the corresponding point P_0; and conversely every sequence in the plane has a well-defined image on the variety, on the $W_{2,3}(\mathsf{P}_0)$ corresponding to its origin P_0. This is true of the unfree sequences as well as the free, for we have seen that an unfree sequence, like a free one, has a well-defined image point on $W_{2,3}$, and hence on the appropriate $W_{2,3}(\mathsf{P}_0)$, i.e. on the variety we have constructed, which is thus a proper model of $W^*_{2,3}$.

The co-ordinates of the image point of an unfree sequence have of course to be found in terms of the invariants of an appropriate singular branch. The procedure is exactly the same as before; we take the co-ordinates (in terms of the affine parametrization) of the image of the branch on $W_{2,3}$, substitute for these their value in terms of the homogeneous parametrization, multiply by a power of $\mathbf{u}^z$, and so obtain those co-ordinates of the image point in which all contravariant indices are z and all covariant indices x or y. The rest are found as before from the equations of the ambient subspace of the appropriate $W_{2,3}(\mathsf{P}_0)$. Using the t-invariants of the singular branches defined in § 21, we find the following image points for the various types of unfree sequence on $W^*_{2,3}$.

Type 2. Cuspidal branch, species 1, (8·3), (21·9), 10:

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_8,\,\alpha_9}_{\beta_1\ldots\beta_5,\,\beta_6} &= -\mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_9}\,\mathbf{q}'_{\beta_1}\ldots\mathbf{q}'_{\beta_6}, \\ Y^{\alpha_1\ldots\alpha_{13}}_{\beta} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_{13}}\,\mathbf{q}'_{\beta}\mathbf{D}'^{*2}, \\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \qquad (22\cdot5)$$

Type 3. Cuspidal branch, species 2, (8·6), (21·9, 11, 12):

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_8,\,\alpha_9}_{\beta_1\ldots\beta_5,\,\beta_6} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_9}\mathbf{q}'_{\beta_1}\ldots\mathbf{q}'_{\beta_6}\mathbf{D}'^*\mathbf{D}''^*,\\ X^{\alpha_1\ldots\alpha_5,\,\alpha_6}_{\beta_1\ldots\beta_8,\,\beta_9} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_6}\mathbf{q}'_{\beta_1}\ldots\mathbf{q}'_{\beta_9}\mathbf{D}''^*,\\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \tag{22·6}$$

Type $\bar{2}$. Specialization of the generic point, with $\mathbf{D}^* = 0$, which makes $\mathbf{G}^{*\alpha}_{\beta} = \mathbf{u}^{\alpha}\mathbf{q}_{\beta}\mathbf{D}^*_{013}$:

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_5,\,\alpha_6}_{\beta_1\ldots\beta_8,\,\beta_9} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_6}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_9}\mathbf{D}^*_{013},\\ Y^{\alpha}_{\beta_1\ldots\beta_{13}} &= \mathbf{u}^{\alpha}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_{13}},\\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \tag{22·7}$$

Type (23). Cubic branch, species 1, (8·4), (21·9):

$$\left.\begin{aligned} Y^{\alpha_1\ldots\alpha_{13}}_{\beta} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_{13}}\mathbf{q}'_{\beta},\\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \tag{22·8}$$

Type 2, 3. Cubic branch, species 2, (8·7), (21·9):

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_8,\,\alpha_9}_{\beta_1\ldots\beta_5,\,\beta_6} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_9}\mathbf{b}'_{\beta_1}\ldots\mathbf{q}'_{\beta_6},\\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \tag{22·9}$$

Type $\bar{2}$, 3. Specialization of (22·6), with $\mathbf{D}'^* = 0$:

$$\left.\begin{aligned} X^{\alpha_1\ldots\alpha_5,\,\alpha_6}_{\beta_1\ldots\beta_8,\,\beta_9} &= \mathbf{u}^{\alpha_1}\ldots\mathbf{u}^{\alpha_6}\mathbf{q}'_{\beta_1}\ldots\mathbf{q}'_{\beta_6},\\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \tag{22·10}$$

Type $\overline{23}$. Specialization of the generic point, with $\mathbf{D}^* = \mathbf{G}^{*\alpha}_{\beta} = 0$:

$$\left.\begin{aligned} Y^{\alpha}_{\beta_1\ldots\beta_{13}} &= \mathbf{u}^{\alpha}\mathbf{q}_{\beta_1}\ldots\mathbf{q}_{\beta_{13}},\\ \text{all other co-ordinates} &= 0. \end{aligned}\right\} \tag{22·11}$$

We have included the separate parametrization of the free sequences of types $\bar{2}$, $\overline{23}$, to show the symmetry of the system. We see from the above parametrizations that the threefold loci, $\Psi_{(23)}$, $\Psi_{2,3}$, $\Psi_{\bar{2},3}$, $\Psi_{\overline{23}}$ are respectively $\mathfrak{w}^{(13,1)}$, $\mathfrak{w}^{(9,6)}$, $\mathfrak{w}^{(6,9)}$, and $\mathfrak{w}^{(1,13)}$; and that the fourfold loci Ψ_2, Ψ_3, $\Psi_{\bar{2}}$, are generated by the line systems joining corresponding points in consecutive pairs of these threefolds.

In exactly the same way, from (7·8), substituting the expressions for all the t-invariants in terms of the coefficients in (21·1) multiplying by $(\mathbf{u}^z)^{81}$, and adjoining all remaining components of the resulting tensors, we obtain the following set of monomials as the parametrization of $W^*_{2,4}$. As they are the components of 70 different tensors, with from 37 to 43 indices each, we do not write them in full as in (22·3), but adopt the notation $(\ \ )^i$, containing a tensor, for the general monomial of degree i in its components, which in the fuller notation would be denoted by writing the tensor i times with different indeterminate indices. With this convention the monomials parametrizing $W^*_{2,4}$ are

$$\left.\begin{aligned} &(\mathbf{u}^{\alpha})^{22+5k-3j}(\mathbf{q}_{\beta})^{10-4k+3j}\mathbf{D}^{*5+3k-j}(\mathbf{G}^{*\alpha}_{\beta})^{1-k}(\mathbf{J}^{*\alpha\gamma}_{\beta\delta})^{1} && (k=0,1;\ j=0,\ldots,5+3k),\\ &(\mathbf{u}^{\alpha})^{25+5k-3j}(\mathbf{q}_{\beta})^{13-4k+3j}\mathbf{D}^{*5+3k-j}(\mathbf{G}^{*\alpha}_{\beta})^{1-k}(\mathbf{I}^{*\alpha\gamma}_{\beta\delta})^{1} && (k=0,1;\ j=0,\ldots,5+3k),\\ &(\mathbf{u}^{\alpha})^{20+5k-3j}(\mathbf{q}_{\beta})^{17-4k+3j}\mathbf{D}^{*1+3k-j}(\mathbf{G}^{*\alpha}_{\beta})^{4-k} && (k=0,\ldots,4;\ j=0,\ldots,1-3k). \end{aligned}\right\} \tag{22·12}$$

The symmetry between the degrees in $(\mathbf{u}^\alpha)$ and $(\mathbf{q}_\beta)$ of the various tensors, which is obvious in (22·1, 2, 3) is not so obvious in this form, but it becomes clear if we tabulate them as follows (in each pair the exponent of $(\mathbf{u}^\alpha)$ is written above, and that of $(\mathbf{q}_\beta)$ below; each line gives the pairs of exponents for different values of j and the values of k shown at the beginning.

Monomials containing $\mathbf{I}^{*\alpha\gamma}_{\beta\delta}$ or $\mathbf{J}^{*\alpha\gamma}_{\beta\delta}$:

$k=0$	28	25	22	19	16	13				with $\mathbf{J}^{*\gamma\alpha}_{\beta\delta}$,
	10	13	16	19	22	25				
	25	22	19	16	13	10				with $\mathbf{I}^{*\alpha\gamma}_{\beta\delta}$,
	13	16	19	22	25	28				
$k=1$	33	30	27	24	21	18	15	12	0	with $\mathbf{J}^{*\alpha\gamma}_{\beta\delta}$,
	6	9	12	15	18	21	24	27	30	
	30	27	24	21	18	15	12	9	6	with $\mathbf{I}^{*\alpha\gamma}_{\beta\delta}$.
	9	12	15	18	21	24	27	30	33	

Monomials containing only $(\mathbf{u}^\alpha)$, $(\mathbf{q}_\beta)$, $\mathbf{D}^*$, $(\mathbf{G}^{*\alpha\gamma}_{\beta\delta})$: (23·1)

$k=0$	20	17												
	17	20												
$k=1$	25	22	19	16	13									
	13	16	19	22	25									
$k=2$	30	27	24	21	18	15	12	9						
	9	12	15	18	21	24	27	30						
$k=3$	35	32	29	26	23	20	17	14	11	8	5			
	5	8	11	14	17	20	23	36	39	32	35			
$k=4$	40	37	34	31	28	25	22	19	16	13	10	7	4	1
	1	4	7	10	13	16	19	22	25	28	31	34	37	40

It is clear that we could go on this way indefinitely, and obtain a parametrization of $W_{2,n}$ for all n, but of course the complication of the results we should obtain increases very rapidly.

23. The dual transformation

It is obvious, as the co-ordinates of the generic point of $W^*_{2,n}$ are all the components of a set of tensors, that every collineation in the plane induces a self-collineation on $W^*_{2,n}$. What is far less obvious, and what we shall now prove, is that every projective duality in the plane, i.e. every point-line correspondence in which the co-ordinates of the image line of a point are linear functions of those of the point, and conversely, likewise induces a self-collineation on $W^*_{2,n}$.

It is sufficient to consider the duality in which a point and line correspond both ways if and only if they have the same co-ordinates. Any algebroid branch has a well-defined dual branch, the co-ordinates of whose generic point are those of the tangent at the generic point of the given branch. If the parametric equations of the given branch are $u^\alpha = f^\alpha(t)$, those of the dual branch can be written

$$\bar{u}^\alpha = \epsilon_{\alpha\beta\gamma} f^\beta(t)\frac{df^\gamma(t)}{dt}$$

(we shall throughout indicate quantities belonging to the dual branch by bars over the letters). If the functions $f^{\alpha}(t)$ are the power series (21·1), this gives a similar expansion for the dual branch

$$\bar{u}^{\alpha} = \bar{u}^{\alpha}_0 + \bar{u}^{\alpha}_1 t + \bar{u}^{\alpha}_2 t^2 + \bar{u}^{\alpha}_3 t^3 + \ldots,$$

where

$$\bar{u}^{\alpha}_i = (i+1)\, q_{\alpha(0,\,i+1)} + (i-1)\, q_{\alpha(1,\,i)} + (i-3)\, q_{\alpha(2,\,i-1)} + \ldots + q_{\alpha(j,\,j+1)} \quad \text{or} \quad 2q_{\alpha(j,\,j+2)} \tag{23·1}$$

according as $i = 2j$ or $i = 2j+1$. Thus in order to determine the series for $\bar{u}^{\alpha}$ as far as the terms in t^n, we need those in the series for u^{α} as far as t^{n+1}. Nevertheless, for the generic branch, all the principal t-invariants of rank $\leqslant n$ of the dual branch are completely determined by the series for the given branch as far as terms in t^n; for we recall that the coefficients u^{α}_n only enter into the t-invariants of rank n by their presence in the determinant D^*_{01n}; and the corresponding determinant for the dual branch is

$$\bar{D}^*_{01n} = \begin{vmatrix} q_{x(01)}, & 2q_{x(02)}, & (n+1)\, q_{x(0,\,n+1)} + (n-1)\, q_{x(1n)} + \ldots \\ q_{y(01)}, & 2q_{y(02)}, & (n+1)\, q_{y(0,\,n+1)} + (n-1)\, q_{y(1n)} + \ldots \\ q_{z(01)}, & 2q_{z(02)}, & (n+1)\, q_{z(0,\,n+1)} + (n-1)\, q_{z(1n)} + \ldots \end{vmatrix}$$

and as

$$u^{\alpha}_0\, q_{\alpha(01)} = u^{\alpha}_0\, q_{\alpha(02)} = u^{\alpha}_0\, q_{\alpha(0,\,n+1)} = 0$$

the first terms in the last column of the determinant are a linear combination of the other two columns, so that the value of $\bar{D}^*_{01n}$, and therefore of any t-invariant of rank $n \geqslant 1$ of the dual branch, is a form in $(u^{\alpha}_0, \ldots, u^{\alpha}_n)$ only. In particular

$$\bar{\mathbf{u}}^{\alpha} = \mathbf{q}_{\alpha}, \quad \bar{\mathbf{q}}_{\alpha} = 2\mathbf{u}^{\alpha}\mathbf{D}^*, \quad \bar{\mathbf{D}}^* = 2\mathbf{D}^{*2}. \tag{23·2}$$

But since $\mathbf{D}^* = 0$ is the condition for the branch to be either singular or inflected, this means that if the given branch is neither singular nor inflected, the dual branch is likewise neither singular nor inflected, and in this case the sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ on the one uniquely determine the sequence $\bar{\mathsf{P}}_0 \ldots \bar{\mathsf{P}}_n$ on the other, i.e. if $\mathsf{P}_0 \ldots \mathsf{P}_n$ is any free sequence with $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ not collinear, there is a uniquely defined similar sequence $\bar{\mathsf{P}}_0 \ldots \bar{\mathsf{P}}_n$ such that all simple branches through the one have as their duals simple branches through the other. This sequence we shall call the dual of the given one.

We cannot immediately say the same thing of a singular or an inflected branch, since the dual of an inflected branch is singular, and that of a singular branch either singular or inflected, and of course the rank of a t-invariant of a singular branch is not very simply related to the suffixes of the coefficients which occur in it. There is, however, one type of sequence for which this can be proved at this stage, and that is what we may call the sequences of maximum condition, for which the proximity conditions, or conditions that some points $\mathsf{P}_2, \ldots$ are collinear with $\mathsf{P}_0\mathsf{P}_1$, uniquely determine all the remaining points given $\mathsf{P}_0\mathsf{P}_1$, i.e. for which all the indices $2 \ldots n$ are present in the condition symbol, with or without bars or brackets.

If the first $q_{\alpha(0i)}$ whose components are not all zero is $q_{\alpha(0r)}$, r is the order of the branch; and if the first D^*_{0rj} which is not zero is $D^*_{0r(r+s)}$, s is called its class; and it is familiar that the dual branch is of order s and class r. If r, s are mutually prime, and $r+s = ar+b$, with $a \geqslant 2$, i.e. $r < s$, and $b < r$, then $\mathsf{P}_0 \ldots \mathsf{P}_a$ are collinear, and all of multiplicity r on the branch except P_a, which is of multiplicity b, whereas on the dual branch $\bar{\mathsf{P}}_0$ is of multiplicity s, and

$\bar{P}_1 \dots \bar{P}_a$ all proximate to it and all of multiplicity r except P_a which is of multiplicity b; and as on both branches P_{a-1}, P_a ($\bar{P}_{a-1}$, $\bar{P}_a$) have multiplicities r, a, which are mutually prime, the succeeding points on the two branches have the same multiplicities and proximity relations, and are all satellites, as far as a certain point P_n, which is simple, and thereafter all points of both branches are simple and free. $P_0 \dots P_n$, $\bar{P}_0 \dots \bar{P}_n$ are thus both sequences of maximum condition, whose type symbols begin with $(2 \dots a)$, $\overline{2 \dots a}$, respectively, and are otherwise the same.

It follows that every sequence $P_0 \dots P_n$ of maximum condition determines uniquely a second sequence $\bar{P}_0 \dots \bar{P}_n$, likewise of maximum condition, such that the dual of every branch through the one passes through the other; each of these sequences will be called the dual of the other.

Since the generic sequence $P_0 \dots P_n$ has a unique dual sequence the relation between the two is mapped by a birational transformation $\mathscr{T}$ of $W^*_{2,n}$ into itself (which for the particular duality in the plane that we are considering is involutory). The base locus of this, i.e. locus of points whose image is not unique, must lie wholly on the Ψ loci, since every point which is on none of these has a unique image, and must consist of the whole of some one or more of these, possibly with different base multiplicities. But as every sequence of maximum condition likewise has a unique image, the base locus of $\mathscr{T}$ cannot contain any of the Ψ loci of maximum condition; and as every Ψ locus contains one or more of those of maximum condition, there is no base locus at all. $\mathscr{T}$ is thus one-one without exception, and every sequence $P_0 \dots P_n$ defines a unique dual sequence $\bar{P}_0 \dots \bar{P}_n$.

Now $W^*_{2,n}$ has on it two nets of primals $|\Gamma|$, $|\Delta|$, given by linear equations in $\mathbf{u}^\alpha$, $\mathbf{q}_\beta$, respectively, each Γ being the locus of images of sequences with origin on a given line, and each Δ of sequences whose tangents pass through a given point. The intersection of two Γ's is an n-fold $\mathfrak{C} = W_{2,n}(P_0)$, locus of images of sequences with a given origin, and the intersection of two Δ's is an n-fold $\mathfrak{D}$, locus of images of sequences with a given tangent. The intersection of $\mathfrak{C}$, $\mathfrak{D}$ is in general empty, but if the point corresponding to $\mathfrak{C}$ lies in the line corresponding to $\mathfrak{D}$ it is a $W_{2,n-1}(P_1)$ on the $W_{2,n}$ $\mathfrak{C}$. These $W_{2,n-1}$'s on $W^*_{2,n}$ we shall denote by W. $W^*_{2,n}$ is in fact a fibre space of the W's over $W^*_{2,1} = \mathfrak{w}^{(1,1)}$, the images in the congruence $\{W\}$ of F, G, f, g on $\mathfrak{w}^{(1,1)}$ being Γ, Δ, $\mathfrak{C}$, $\mathfrak{D}$.

$\mathscr{T}$ obviously interchanges $|\Gamma|$ with $|\Delta|$, and $\{\mathfrak{C}\}$ with $\{\mathfrak{D}\}$, and transforms each W into a W—linearly, since the base on $W_{2,n-1}$ is so unsymmetrical that it obviously has no linear system on it with the same properties as the prime sections, except the prime sections. The linear system on $\mathfrak{C}$ which is transformed into the prime sections of $\mathfrak{D}$ thus traces the prime sections on each of the pencil $|W|$ on $\mathfrak{C}$, and so can differ from the prime sections of $\mathfrak{C}$ at most by a multiple of $|W|$. But if the prime sections of $\mathfrak{D}$ were the images of those of $\mathfrak{C}$ together with $|hW|$, the image on $\mathfrak{D}$ of the line l_1 on $\mathfrak{C}$, unisecant to $|W|$, would be a curve of degree $h+1$, and there would be no curve on $\mathfrak{D}$, unisecant to $|W|$, of lower order than this. Now one tensor in the parametrization of $W^*_{2,n}$ is, in the notation of (22·12)

$$(\mathbf{u}^\alpha)^1 \, (\mathbf{q}_\beta)^{\frac{1}{2}(3^n-1)},$$

and as the collinearity of $P_0 \dots P_n$ is given by the vanishing of all the t-invariants of rank $\leqslant n$ except $\mathbf{u}^\alpha$, $\mathbf{b}_\beta$, $\Psi_{\overline{2\dots n}}$ is parametrized by this tensor only, all the other co-ordinates vanishing. Putting constant values of $\mathbf{q}_\beta$ into this tensor we see that $\Psi_{\overline{2\dots n}}$ traces on $\mathfrak{D}$ a line unisecant to the pencil $|W|$; thus $h = 0$, $\mathfrak{D}$ is a projective image of $\mathfrak{C}$, and $\mathscr{T}$ maps them projectively on each other.

Hence in the same way, the mapping system of $\mathscr{T}$ must differ from the prime sections of $W^*_{2,n}$ by something which has zero intersection with $\mathfrak{C}$, and since $\mathscr{T}$ is involutory, this system is also the mapping system for $\mathscr{T}^{-1}$, and must similarly differ from the prime sections by something having zero intersection with $\mathfrak{D}$. But as on $\mathfrak{w}^{(1,1)}$ there is no linear system except $|0|$ having no intersection with either $\{f\}$ or $\{g\}$, there is none on $W^*_{2,n}$ having zero intersection with both $|\mathfrak{C}|$ and $|\mathfrak{D}|$. Thus the mapping system of $\mathscr{T}$ does not differ from the prime sections, i.e. $\mathscr{T}$ is a collineation.

This can be verified easily in the cases $n = 1, 2, 3$. From (23·1) we have, as well as (23·2),

$$\bar{q}_{\alpha(02)} = 3u^\alpha_0 D^*_{013} + u^\alpha_1 D^*_{012}, \quad \bar{D}_{013} = 4D^*_{012} D^*_{013},$$

so that

$$\bar{\mathbf{G}}^{*\beta}_\alpha = -4\mathbf{D}^{*2}\mathbf{G}^{*\alpha}_\beta. \tag{23·3}$$

Substituting from (23·2) in (22·1) we have

$$\bar{X}^\alpha_\beta = 2\mathbf{D}^* X^\beta_\alpha$$

in (22·2)

$$\bar{X}^{\alpha\beta\gamma\delta}_\epsilon = 2\mathbf{D}^{*3} Y^\epsilon_{\alpha\beta\gamma}, \quad \bar{Y}^\epsilon_{\alpha\beta\gamma\delta} = 2^4\mathbf{D}^{*3} X^{\alpha\beta\gamma\delta}_\epsilon,$$

and from (23·2, 3) in (22·3)

$$\bar{X}^{\alpha_1\ldots\alpha_8,\gamma}_{\beta_1\ldots\beta_5,\delta} = -2^7\mathbf{D}^{*9} X^{\beta_1\ldots\beta_5,\delta}_{\alpha_1\ldots\alpha_8,\gamma}, \quad \bar{X}^{\alpha_1\ldots\alpha_5,\gamma}_{\beta_1\ldots\beta_8,\delta} = -2^{10}\mathbf{D}^{*9} X^{\beta_1\ldots\beta_8,\delta}_{\alpha_1\ldots\alpha_5,\gamma},$$
$$\bar{Y}^{\alpha_1\ldots\alpha_{13}}_\beta = 2\mathbf{D}^{*9} Y^\beta_{\alpha_1\ldots\alpha_{13}}, \quad \bar{Y}^\alpha_{\beta_1\ldots\beta_{13}} = 2^{13}\mathbf{D}^{*9} Y^{\beta_1\ldots\beta_{13}}_\alpha,$$
$$\bar{Y}^{\alpha_1\ldots\alpha_{10}}_{\beta_1\ldots\beta_4} = 2^4\mathbf{D}^{*9} Y^{\beta_1\ldots\beta_4}_{\alpha_1\ldots\alpha_{10}}, \quad \bar{Y}^{\alpha_1\ldots\alpha_4}_{\beta_1\ldots\beta_{10}} = 2^{10}\mathbf{D}^{*9} Y^{\beta_1\ldots\beta_{10}}_{\alpha_1\ldots\alpha_4},$$
$$\bar{Y}^{\alpha_1\ldots\alpha_7}_{\beta_1\ldots\beta_7} = 2^7\mathbf{D}^{*9} Y^{\beta_1\ldots\beta_7}_{\alpha_1\ldots\alpha_7},$$

as the effects of the transformation $\mathscr{T}$ on $W^*_{2,1}$, $W^*_{2,2}$, $W^*_{2,3}$. It is clear that the effect on $W^*_{2,n}$ for all n is of the same kind; it merely interchanges (with a multiplier $\mathbf{D}^{*3^{n-1}}$ and some numerical coefficients) the co-ordinate tensors by pairs, raising all the covariant and lowering all the contravariant indices.

This enables us to obtain the values of $\bar{\mathbf{I}}^{*\alpha\beta}_{\gamma\delta}$, $\bar{\mathbf{J}}^{*\alpha\beta}_\gamma$ without substitution in the lengthy expressions (21·6). On looking at the table (22·13) of the exponents of $\mathbf{u}^\alpha$, $\mathbf{q}_\beta$ in the parametrization of $W^*_{2,4}$ we see that two pairs of co-ordinate tensors which must be interchanged in this way are

$$(\mathbf{u}^\alpha)^{28}(\mathbf{q}_\beta)^{10}\mathbf{D}^{*5}(\mathbf{G}^{*\alpha}_\beta)^1(\mathbf{J}^{*\alpha\gamma}_{\beta\delta})^1, \quad (\mathbf{u}^\alpha)^{10}(\mathbf{q}_\beta)^{28}(\mathbf{G}^{\alpha}_\beta{}^*)^1(\mathbf{I}^{*\alpha\gamma}_{\beta\delta})^1,$$
$$(\mathbf{u}^\alpha)^{33}(\mathbf{q}_\beta)^6\mathbf{D}^8(\mathbf{J}^{*\alpha\gamma}_\beta)^1, \quad (\mathbf{u}^\alpha)^6(\mathbf{q}_\beta)^{33}(\mathbf{I}^{*\alpha\gamma}_{\beta\delta})^1$$

and this can only be the case if

$$\bar{\mathbf{I}}^{*\alpha\gamma}_{\beta\delta} = -4\mathbf{D}^{*2}\mathbf{J}^{*\beta\delta}_{\alpha\gamma}, \quad \bar{\mathbf{J}}^{*\alpha\gamma}_{\beta\delta} = -8\mathbf{D}^{*6}\mathbf{I}^{*\beta\delta}_{\alpha\gamma}.$$

(The numerical coefficients -4, -8 follow from the fact that both the unbarred and barred t-invariants satisfy the identity (21·7).)

24. The Ψ loci on $W^*_{2,n}$

The relationship between the various Ψ loci on $W^*_{2,n}$ can best be studied by considering their traces on the general W, which of course are the same as the traces on $W_{2,n-1}(\mathsf{P}_1)$ of the corresponding Φ loci on $W_{2,n}$, since the trace of each Ψ locus on $\mathfrak{C}$ is the corresponding

Φ locus on this $W_{2,n}$. The investigation of § 11 are applicable here. The Ψ loci of maximum condition, 2^{n-1} in number, each trace a point on each W, and each is consequently a $\mathfrak{w}^{(i,j)}$ for some values of (i,j), and is represented by the vanishing of all the co-ordinates in the parametrization except those of one tensor, with i contravariant and j covariant indices, which for the corresponding singular branch reduces to $(\mathbf{u}'^{\alpha})^i(\mathbf{q}'_\beta)^j$. We have seen in § 11 how the points traced by these on W are arranged in a sequence, joined consecutively by lines which are the traces of ruled Φ surfaces of $W_{2,n}$, and hence of four-dimensional Ψ loci of $W^*_{2,n}$; and it is obvious that the duality transformation $\mathscr{T}$ of the last section simply turns the whole sequence end-to-end, and in fact interchanges the right and left halves of each of figures 2 to 4, and the other figures that could be constructed in the same way for higher values of n, in a mirror reflexion. Incidentally we see that, as was established directly in the last section for sequences of maximum condition, so quite generally the condition symbols of two sequences dual to each other are interchanged, by interchanging $(2\ldots a)$ with $\overline{2\ldots a}$ at the beginning, and leaving the rest unaltered.

If on each of these $\mathfrak{w}^{(i,j)}$'s we denote the images of F, G, f, g by F, G, f, g with the appropriate suffixes, we see that $\Gamma, \Delta, \mathfrak{C}, \mathfrak{D}$ trace the corresponding F, G, f, g on each of them, and in particular that the curves f (of order j) are the corresponding Φ curves on $\mathfrak{C}$. The orders of these curves have been found for $n \leqslant 5$ in § 11, which gives us the values of j; those of i are of course the same sequence of integers in the reverse order. We have in fact seen already from the detailed parametrization that the two Ψ loci on $W^*_{2,2}$ are $\mathfrak{w}^{(4,1)}$, $\mathfrak{w}^{(14)}$, and that the four threefold Ψ loci on $W^*_{2,3}$ are $\mathfrak{w}^{(13,1)}$, $\mathfrak{w}^{(9,6)}$, $\mathfrak{w}^{(6,9)}$, $\mathfrak{w}^{(1,13)}$; and from the orders of the Φ curves on $W_{2,4}$, found in § 11, we find the following values of i, j for $W^*_{2,4}$.

Condition:	(234)	(23) 4	2, 3, 4	2(34)	$\overline{2}$(34)	$\overline{2}$, 3, 4	$\overline{23}$, 4	$\overline{234}$
i:	40	35	31	24	21	13	8	1
j:	1	8	13	21	24	31	35	40

and it is not hard to identify the co-ordinate tensors in (22·12) that have the right number of indices; they are (in the above order)

$$(\mathbf{u}^\alpha)^{40}(\mathbf{q}_\beta)^1\mathbf{D}^{*13}, \quad (\mathbf{u}^\alpha)^{33}(\mathbf{q}_\beta)^6\mathbf{D}^{*8}(\mathbf{J}^{*\alpha\gamma}_{\beta\delta})^1,$$
$$(\mathbf{u}^\alpha)^{28}(\mathbf{q}_\beta)^{10}\mathbf{D}^{*5}(\mathbf{G}^{*\alpha}_\beta)^1(\mathbf{J}^{*\alpha\gamma}_{\beta\delta})^1, \quad (\mathbf{u}^\alpha)^{20}(\mathbf{q}_\beta)^{17}\mathbf{D}^*(\mathbf{G}^{*\alpha}_\beta)^4,$$
$$(\mathbf{u}^\alpha)^{17}(\mathbf{q}_\beta)^{20}(\mathbf{G}^{*\alpha}_\beta)^4, \quad (\mathbf{u}^\alpha)^{10}(\mathbf{q}_\beta)^{28}(\mathbf{G}^{*\alpha}_\beta)^1(\mathbf{I}^{*\alpha\gamma}_{\beta\delta})^1,$$
$$(\mathbf{u}^\alpha)^6(\mathbf{q}_\beta)^{33}(\mathbf{I}^{*\alpha\beta}_{\gamma\delta})^1, \quad (\mathbf{u}^\alpha)^1(\mathbf{q}_\beta)^{40}.$$

Similarly the values of j for the sixteen three-dimensional Ψ loci on $W^*_{2,5}$ are

1, 10, 17, 29, 34, 47, 55, 66, 69, 79, 86, 97, 101, 110, 115, 121,

and those of i are the same in the reverse order.

Each of the four-dimensional Ψ loci traces either a line or a curve on W, joining the point traces of two of the three-dimensional Ψ loci; it is therefore generated by lines or curves joining corresponding points of these, say $\mathfrak{w} = \mathfrak{w}^{(i,j)}$, $\mathfrak{w}' = \mathfrak{w}^{(i',j')}$. Now the theory of the fourfold loci $\mathfrak{B}^{(i,j)(i',j')}$ generated by lines joining the corresponding points of $\mathfrak{w} = \mathfrak{w}^{(i,j)}$, $\mathfrak{w}' = \mathfrak{w}^{(i+h,j+k)}$ is very simple; defining $\mathfrak{s}$, $\mathfrak{t}$, S, T as in § 19 and denoting the generating lines by p, using (19·5), the whole intersection table is easily found (table 7), all the intersections being obvious except the sections by $\mathfrak{w}, \mathfrak{w}'$ of themselves and their subvarieties, and these follow from the fact that the corresponding sections by $\mathfrak{w}'$, $\mathfrak{w}$, respectively, are zero, together with the linear identity between the four columns. All the intersections depend only on h, k and not at all on i, j; in fact it is clear that for any i, j, on the projective model of the system $|\mathfrak{w}+(i+h)\,\mathfrak{s}+(j+k)\,\mathfrak{t}|$, $\mathfrak{w}$, $\mathfrak{w}'$ appear as $\mathfrak{w}^{(i,j)}$, $\mathfrak{w}^{(i+h,j+k)}$, respectively. Further, on that of the system $|m\mathfrak{w}+(i+mh)\,\mathfrak{s}+(j+mk)\,\mathfrak{t}|$ $\mathfrak{w}$, $\mathfrak{w}'$ appear as $\mathfrak{w}^{(i,j)}$, $\mathfrak{w}^{(i+mh,j+mk)}$, and $\{p\}$ as a congruence of curves of order m. Conversely, any fourfold generated by an ∞^3 congruence of rational curves to which a $\mathfrak{w}^{(i,j)}$ and a $\mathfrak{w}^{(i',j')}$ (not intersecting each other) are unisecant is of this kind, since it contains surfaces F generated by a pencil in the congruence with unisecant curves f, f', whose difference is accordingly a multiple of the congruence $\{p\}$, and similarly for the curves g, g'; this establishes the values of h, k and the linear identity between the columns of the intersection table. We note that $i'-i$, $j'-j$ are

TABLE 7

	$\mathfrak{w}$	$\mathfrak{s}$	$\mathfrak{t}$	$\mathfrak{w}'=\mathfrak{w}+h\mathfrak{s}+k\mathfrak{t}$
$\mathfrak{w}$	$-hF-kG$	F	G	0
$\mathfrak{s}$	F	S	$S+T$	$F'=F+(h+k)\,S+kT$
$\mathfrak{t}$	G	$S+T$	T	$G'=G+hS+(h+k)\,T$
$\mathfrak{w}'$	0	F'	G'	$hF'+kG'$
F	$-(h+k)\,f-kg$	f	$f+g$	0
G	$-hf-(h+k)\,g$	$f+g$	g	0
S	f	0	p	$f'=f+kp$
T	g	p	0	$g'=g+hp$
F'	0	f'	$f'+g'$	$(h+k)\,f'+kg'$
G'	0	$f'+g'$	g'	$hf'+(h+k)\,g'$
f	$-k$	0	1	0
g	$-h$	1	0	0
p	1	0	0	1
f'	0	0	1	k
g'	0	1	0	h

always multiples of m (the quotients being h, k) and that on the minimum order model $\mathfrak{w}$, $\mathfrak{w}'$ appear as $\mathfrak{w}^{(1,1)}$, $\mathfrak{w}^{(h+1,k+1)}$ if h, k are both positive, but as $\mathfrak{w}^{(1-h,1)}$, $\mathfrak{w}^{(1,k+1)}$ if h is negative and k positive; a qualitative difference between the two cases is that in the former $\mathfrak{w}$ is unique but $\mathfrak{w}'$ varies in a linear system, whereas in the latter both are unique. The latter case is that which we chiefly encounter in this work, but we note that the locus $\Phi_{2,3}$ on $W_{3,3}$ has $h=1$, $k=3$; $\Phi_{(23)3}$ is unique, but $\Phi_{2,3,\bar{3}}$ is transformed into a system of equivalent loci by the collineations induced by regular transformations in S_3.

All the fourfold Ψ loci on $W^*_{2,n}$ are generated in this way; we can tabulate the values of m, i, j, i', j', h, k for all those on $W^*_{2,3}$, $W^*_{2,4}$, $W^*_{2,5}$ from the information in § 11 as given in table 8.

Comparing the values of h, k in the last column, we see that for each value of n, the conditions which include n in the symbol (i.e. for which P_n is a satellite) give Ψ loci which are birationally equivalent to those on $W_{2,n-1}$ corresponding to the same symbol with n omitted (whether it is in a bracket or not). This is to be expected, as every sequence $\mathsf{P}_0 \ldots \mathsf{P}_{n-1}$ of the given type determines a unique sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ by adding the point P_n in the satellite position (or in either of the two satellite positions if P_{n-1} is itself a satellite).

Moreover, on $W^*_{2,4}$, $\Psi_{(34)}$ and $\Psi_{3,4}$ being birational images of each other (and of Ψ_3 on $W^*_{2,3}$, and of $W^*_{2,2}$ itself) we see from figure 4 that Ψ_3 is generated by lines joining corresponding points of $\Psi_{(34)}$, $\Psi_{3,4}$. Several of the fivefold Ψ loci on $W^*_{2,5}$, and quite a lot on $W^*_{2,6}$, are generated in this way; on $W^*_{2,5}$ $\Psi_{2(45)}$, $\Psi_{2,4,5}$ are birationally equivalent (with $h, k = -4, 5$) and $\Psi_{2,4}$ is generated by lines joining corresponding points; and $\Psi_{\bar{2},4}$ similarly from $\Psi_{\bar{2}(45)}$, $\Psi_{\bar{2},4,5}$. Moreover $\Psi_{(345)}$, $\Psi_{(34)5}$, $\Psi_{3,4,5}$, and $\Psi_{3(45)}$ are all birationally equivalent (with $h, k = -3, 3$) and the lines joining their corresponding points consecutively generate $\Psi_{(34)}$, $\Psi_{3,5}$, $\Psi_{3,4}$. On $W^*_{2,6}$ we see in figure 4 that $\Psi_{(3456)}$, $\Psi_{(345)6}$, $\Psi_{(34)5,6}$, $\Psi_{(34)(56)}$, $\Psi_{3,4(56)}$, $\Psi_{3,4,5,6}$, $\Psi_{3(45)6}$, $\Psi_{3(456)}$ are all birationally equivalent, and the lines joining corresponding points consecutively generate $\Psi_{(345)}$, $\Psi_{(34)6}$, $\Psi_{(34)5}$, $\Psi_{3(56)}$, $\Psi_{3,4,5}$, $\Psi_{3,4,6}$, $\Psi_{3(45)}$;

Table 8

	condition	m	i	j	i'	j'	h	k
$W_{2,3}$	2	1	13	1	9	6	-4	5
	3	1	9	6	6	9	-3	3
	$\bar{2}$	1	6	9	1	13	-5	4
$W_{2,4}$	(23)	1	40	1	35	8	-5	7
	2, 4	1	35	8	31	13	-4	5
	2, 3	1	31	13	24	21	-7	8
	(34)	1	24	21	21	24	-3	3
	$\bar{2}$, 3	1	21	24	13	31	-8	7
	$\bar{2}$, 4	1	13	31	8	35	-5	4
	$\overline{23}$	1	8	35	1	40	-7	5
	3, 4	6	31	13	13	31	-3	3
$W_{2,5}$	(234)	1	121	1	115	10	-6	9
	(23) 5	1	115	10	110	17	-5	7
	(23) 4	1	110	17	101	29	-9	12
	2(45)	1	101	29	97	34	-4	5
	2, 3, 4	1	97	34	86	47	-11	13
	2, 3, 5	1	86	47	79	55	-7	8
	2(34)	1	79	55	69	66	-10	11
	(345)	1	69	66	66	69	-3	3
	$\bar{2}$(34)	1	66	69	55	79	-11	10
	$\bar{2}$, 3, 5	1	55	79	47	86	-8	7
	$\bar{2}$, 3, 4	1	47	86	34	97	-13	11
	$\bar{2}$(45)	1	34	97	29	101	-5	4
	$\overline{23}$, 4	1	29	101	17	110	-12	9
	$\overline{23}$, 5	1	17	110	10	115	-7	5
	$\overline{234}$	1	10	115	1	121	-9	6
	2, 4, 5	6	110	17	86	47	-4	5
	$\bar{2}$, 4, 5	6	47	86	17	110	-5	4
	(34) 5	8	79	55	55	79	-3	3
	3, 4, 5	13	86	47	47	86	-3	3
	3(45)	21	97	34	34	97	-3	3

and there are ten more similar structures, five on Ψ_2 and five on $\Psi_{\bar{2}}$; we have also $\Psi_{3,5,6}$ generated by a congruence of ∞^4 sextic curves, the images of $\Phi_{2,3}$ on each of the ∞^4 subsystem of $\{W_{2,3}(\mathsf{P}_3)\}$ that generates Ψ_3, which join corresponding points of $\Psi_{(34)5,6}$, $\Psi_{3,4,5,6}$.

There are other fivefold Ψ loci generated by an ∞^4 line or curve congruence, to which one of the fourfold Ψ loci is unisecant and containing two ∞^3 subsystems (meeting the unisecant fourfold in its $\mathfrak{w}$, $\mathfrak{w}'$) which generate two other fourfold Ψ loci; obvious examples are Ψ_2, $\Psi_{\bar{2}}$ on $W^*_{2,4}$, and the many which appear in figures 3 and 4 as ruled surfaces having a minimum directrix line and two generators among the lines of the figure; and we have also for instance $\Psi_{4,5}$ on $W^*_{2,5}$, generated by an ∞^4 congruence of sextic curves, of which two ∞^3 subsystems generate $\Psi_{2,4,5}$, $\Psi_{\bar{2},4,5}$. Of these there is little to be said except that where a second unisecant fourfold, not meeting the first, is not apparent, we cannot assume its existence, though many of the properties of the fivefold locus are the same as if it did exist. For instance, it might perhaps be tempting to suppose that on the fivefold locus Ψ_4 on $W^*_{2,4}$, which is generated by ∞^4 lines unisecant to $\Psi_{3,4}$, including two ∞^3 subsystems generating $\Psi_{2,4}$, $\Psi_{\bar{2},4}$, there might be a second fourfold locus, unisecant to the generating lines, birationally equivalent to $\Psi_{3,4}$, generated by ∞^3 rational curves of order nine, joining corresponding points of $\Psi_{(34)4}$, $\Psi_{\overline{23},4}$; Ψ_4 would then be generated in the same way as Ψ_3, by the lines joining corresponding points of these birationally equivalent fourfolds, and the generating curves of the second fourfold would be a second directrix curve of each of the ruled surfaces (of order 15) traced on Ψ_4 by $\{W\}$, of which one appears as the trace of Ψ_4 in figure 2. All the loci are of the orders one would expect if this were so, and the equivalences on Ψ_4 and its sub Ψ loci which one would deduce from it in fact hold. All this, however, is not the case; Ψ_4 on $W^*_{2,4}$ is a birational model of $W^*_{2,3}$, and we shall see in the next section that there is no primal on $W^*_{2,3}$, unisecant to its generating lines and not meeting Ψ_3.

Of the Ψ loci of more than five dimensions it is not easy to say anything descriptive without more detailed study, except that all of them are fibre spaces of lines over those of lower dimensions, or fibre spaces of rational curves which are birational images of fibre spaces of lines. If in fact Ψ is any Ψ locus on $W^*_{2,n}$, there is at least one on $W^*_{2,n+1}$ which is a birational image of Ψ, whose condition symbol is obtained by adding $n+1$ at the end of that of Ψ; and if the latter contains n, we can obtain two such images by adding $n+1$ either bracketed or unbracketed; and the Ψ locus on $W^*_{2,n+1}$ which has the same condition symbol as Ψ is generated by lines l_{n+1} unisecant to this birational image of Ψ, or joining corresponding points of its two images.

25. Base and intersection theory on $W^*_{2,n}$

$W^*_{2,2}$ being $\mathfrak{V}^{(4,1)(1,4)}$ its intersection and equivalence relations are those tabulated in the last section with $h, k = -3, 3$. The whole theory of this fourfold was found by Gherardelli, using only the known properties of $\mathfrak{w}^{(i,j)}$ and the relation

$$\Psi_2 - 3\Gamma = \Psi_{\bar{2}} - 3\Delta \tag{25·1}$$

which is $\mathfrak{w} + h\mathfrak{s} = \mathfrak{w}' - k\mathfrak{t}$ of the last section, and which he obtained from the fact that if a plane curve of order n and class m has k cusps and i inflexions, $k - 3n = i - 3m$. (25·1) is valid on every $W^*_{2,n}$, $n \geq 2$; since $W^*_{2,n}$ is a fibre space of lines over $W^*_{2,n-1}$, and the images in this line congruence of Γ, Δ, Ψ_2, $\Psi_{\bar{2}}$ on $W^*_{2,n-1}$ are the similarly defined loci on $W^*_{2,n}$. We shall in every case obtain a more symmetrical base by using, in place of either Ψ_2 or $\Psi_{\bar{2}}$, the linear system

$$\Theta = \Psi_2 + 3\Delta = \Psi_{\bar{2}} + 3\Gamma.$$

We take then on $W^*_{2,2}$ Γ, Δ, Θ as a base for primals, $I = F_2$, $J = G_{\bar{2}}$, S, T as a base for surfaces, and $l_0 = f_2$, $l_1 = g_{\bar{2}}$, $l_2 = p$ as a base for curves; where S, T are the ruled quintic $W_{2,2}(\mathsf{P}_0)$ and its dual image (i.e. its transform under $\mathscr{T}$ or the collineation induced by any other duality in the plane), which we called $\mathfrak{C}$, $\mathfrak{D}$ in the general case; and I, J are surfaces ruled in $\{l_0\}$, $\{l_1\}$, both of order 9, and correspond to the similarly named surfaces on the locus $\Phi_{(23)}$ on $W_{3,3}$, which we recall is a projective image of $W^*_{2,2}$. We have the following equivalences:

$$\left.\begin{aligned} \Psi_2 &= \Theta - 3\Delta, \quad G_2 = J + 3S, \quad g_2 = l_1 + 3l_2, \\ \Psi_{\bar{2}} &= \Theta - 3\Gamma, \quad F_{\bar{2}} = I + 3T, \quad f_{\bar{2}} = l_0 + 3l_2, \end{aligned}\right\} \qquad (25\cdot 2)$$

and the intersection table (table 9).

We notice that on every $W^*_{2,n}$, as $\Psi_{(2\ldots n)} = \mathfrak{w}^{(m_n, 1)}$ and $\Psi_{\overline{2\ldots n}} = \mathfrak{w}^{(1, m_n)}$, where $m_n = \frac{1}{2}(3^n - 1)$, we have two ∞^2 line systems $\{l_0\} = \{f_{(2\ldots n)}\}$, $\{l_1\} = \{g_{\overline{2\ldots n}}\}$; the former contains the unique line on each $\mathfrak{C}$, image of l_1 on $W_{2,n}$, and the latter the corresponding line on each $\mathfrak{D}$. Further, we have for $i = 2, \ldots, n-1$ the ∞^{i+1} line system $\{l_i\}$ generating $\Psi_{(i+1\ldots n)}$, and the ∞^{n+1} line system $\{l_n\}$ generating $W^*_{2,n}$ itself; the subsystem of $\{l_i\}$ $(i = 2, \ldots, n)$ on each $\mathfrak{C}$ or $\mathfrak{D}$ being the image of $\{l_i\}$ on $W_{2,n}$, in particular $\{l_2\}$ contributes to each W the unique line seen as a horizontal line in the upper centre of figures 2 to 4, and the vertical lines in these figures are all in $\{l_n\}$. We have also on each $W^*_{2,n}$ $(n \geqslant 2)$ the surfaces $I = F_{(2\ldots n)}$, $J = G_{\overline{2\ldots n}}$, of order 3^n, ruled in $\{l_0\}$, $\{l_1\}$, respectively.

TABLE 9

$W^*_{2,2}$	Π	Δ	Θ	$\Pi = \Gamma + \Delta + \Theta$	330
Γ	S	$S+T$	$I+3(S+T)$	$I+5S+4T$	54
Δ	$S+T$	T	$J+3(S+T)$	$J+4S+5T$	54
Θ	$I+3(S+T)$	$J+3(S+T)$	$3(I+J)+9(S+T)$	$4(I+J)+15(S+T)$	222
I	l_0	$l_0+l_1+3l_2$	$3l_0$	$5l_0+l_1+3l_2$	9
J	$l_0+l_1+3l_2$	l_1	$3l_1$	$l_0+5l_1+3l_2$	9
S	0	l_2	l_0+3l_2	l_0+4l_2	5
T	l_2	0	l_1+3l_2	l_1+3l_2	5
l_0	0	1	0	1	1
l_1	1	0	0	1	1
l_2	0	0	1	1	1

Turning now specifically to $W^*_{2,3}$, as $W_{2,3}(\mathsf{P}_0)$ and its dual image are threefolds, we shall denote them by $\mathfrak{c}$, $\mathfrak{d}$, rather than $\mathfrak{C}$, $\mathfrak{D}$. As $W^*_{2,3}$ is a fibre space of lines $\{l_3\}$ over $W^*_{2,2}$, and Ψ_3 is unisecant to this congruence, every subvariety of $W^*_{2,2}$ has on $W^*_{2,3}$ an image in $\{l_3\}$ and another image on Ψ_3, the latter being of course the intersection of the former with Ψ_3; and these two images of each element of a base (of all dimensions) on $W^*_{2,2}$ form together a base of all dimensions on $W^*_{2,3}$. These images are as follows:

image of $W^*_{2,2}$	in $\{l_3\}$ $W^*_{2,3}$	on Ψ_3 Ψ_3
Γ	Γ	$\mathfrak{s}_3$
Δ	Δ	$\mathfrak{t}_3$
Θ	Θ	$\mathfrak{u}$
I	$\mathfrak{s}_2$	$F_{2,3}$
J	$\mathfrak{t}_{\bar{2}}$	$G_{\bar{2},3}$

image of	in $\{l_3\}$	on Ψ_3
S	$\mathfrak{c}$	S_3
T	$\mathfrak{d}$	T_3
l_0	S_2	$f_{2,3}$
l_1	$T_{\bar{2}}$	$g_{\bar{2},3}$
l_2	W	l_2
point	l_3	point

The only new notation is $\mathfrak{u} = \Psi_{2,3} + 3\mathfrak{t}_3 = \Psi_{\bar{2},3} + 3\mathfrak{s}_3$, the linear system traced by Θ on Ψ_3. From (25·2) applied to $\{l_3\}$

$$\left.\begin{aligned} \Psi_2 &= \Theta - 3\Delta, \quad \mathfrak{t}_2 = \mathfrak{t}_{\bar{2}} + 3\mathfrak{c}, \quad T_2 = T_{\bar{2}} + 3W, \\ \Psi_{\bar{2}} &= \Theta - 3\Gamma, \quad \mathfrak{s}_{\bar{2}} = \mathfrak{s}_2 + 3\mathfrak{d}, \quad S_{\bar{2}} = S_2 + 3W. \end{aligned}\right\} \qquad (25\cdot 3)$$

From the geometry on Ψ_2, Ψ_3, $\Psi_{\bar{2}}$, using the values of h, k from table 8:

$$\left.\begin{aligned} \Psi_{2,3} &= \mathfrak{u}-3\mathfrak{t}_3, & \Psi_{(23)} &= \mathfrak{u}-3\mathfrak{t}_3+4\mathfrak{s}_2-5(\mathfrak{t}_{\bar{2}}+3\mathfrak{c}),\\ \Psi_{\bar{2},3} &= \mathfrak{u}-3\mathfrak{s}_3, & \Psi_{\bar{2}\bar{3}} &= \mathfrak{u}-3\mathfrak{s}_3-5(\mathfrak{s}_2+3\mathfrak{d})+4\mathfrak{t}_{\bar{2}} \end{aligned}\right\} \tag{25·4}$$

and

$$\left.\begin{aligned} F_{(23)} &= I, & G_{\bar{2}\bar{3}} &= J,\\ F_{2,3} &= I+S_2+5(T_{\bar{2}}+3W), & G_{\bar{2},3} &= J+5(S_2+3W)+T_{\bar{2}},\\ F_{\bar{2},3} &= F_{2,3}+3T_3, & G_{2,3} &= G_{\bar{2},3}+3S_3,\\ F_{\bar{2}\bar{3}} &= F_{\bar{2}}-(S_2+3W)+4T_{\bar{2}} & G_{(23)} &= G_{2,3}+4S_2-(T_{\bar{2}}+3W)\\ &= I+3T_3+9T_{\bar{2}}+12W, & &= J+9S_2+3S_3+12W. \end{aligned}\right\} \tag{25·5}$$

Finally, from $f_{(23)} = l_0$, $g_{\bar{2}\bar{3}} = l_1$, using equivalences on the ruled surfaces S, T

$$\left.\begin{aligned} f_{2,3} &= l_0+5l_3, & f_{\bar{2},3} &= l_0+3l_2+5l_3, & f_{\bar{2}\bar{3}} &= l_0+3l_2+9l_3,\\ g_{\bar{2},3} &= l_1+5l_3, & g_{2,3} &= l_1+3l_2+5l_3, & g_{(23)} &= l_1+3l_2+9l_3. \end{aligned}\right\} \tag{25·6}$$

Thus everything has been expressed in terms of base which we can take to consist of the primals Γ, Δ, Θ, Ψ_3, threefolds $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{u}$, $\mathfrak{s}_2$, $\mathfrak{t}_{\bar{2}}$, $\mathfrak{s}_3$, $\mathfrak{t}_3$, surfaces I, J, W, S_2, $T_{\bar{2}}$, S_3, T_3, and lines l_0, l_1, l_2, l_3. In the intersection table (Table 10) however, a number of the entries are simplified by writing $F_{2,3}$, $G_{\bar{2},3}$ in place of their values in terms of the base as given by (25·5).

As Ψ_3 is a birational image of $W^*_{2,2}$, on which the lines l_0, l_1, l_2 of the latter appear as the sextic curves $f_{2,3}$, $g_{\bar{2},3}$, and the lines l_2, it follows that Ψ_3 is the projective model of the system $6(\Gamma+\Delta)+\Theta$ on $W^*_{2,2}$, i.e. its prime sections are traced on it by the similarly denoted system on $W^*_{2,3}$; and if as usual we denote the prime sections of $W_{2,3}$ by $|\Pi|$, this means that

$$\Pi\,.\,\Psi_3 = 6(\mathfrak{s}_3+\mathfrak{t}_3)+\mathfrak{u}. \tag{25·7}$$

On the other hand the residual section of $W^*_{2,3}$ by a prime through Ψ_3 is compounded with $\{l_3\}$, and as it has one generator in common with each of S_2, $T_{\bar{2}}$, and four with W,

$$\Pi = \Gamma+\Delta+4\Theta+\Psi_3. \tag{25·8}$$

The intersection table is now easily written down. Most of the intersections with Γ, Δ, Θ are immediate consequences of the corresponding intersections on $W^*_{2,2}$, applied either to $\{l_3\}$ or to Ψ_3; the intersections of Ψ_3 with itself and its subvarieties are most simply obtained by cutting (25·8) by Ψ_3 on both sides and comparing with (25·7), which gives

$$\Psi_3\,.\,\Psi_3 = 5(\mathfrak{s}_3+\mathfrak{t}_3)-3\mathfrak{u},$$

which means that the virtual intersection of any subvariety of Ψ_3 with Ψ_3 is the same as its intersection with the virtual system of primals $|5(\Gamma+\Delta)-3\Theta|$.

We could obtain a variety on which all the equivalences (25·3, ..., 8) and the intersection relations given in table 10 would be valid by taking, in an ambient skew to that of Ψ_3, a second birational image Ψ^* of $W^*_{2,2}$, on which the generators $\{l_2\}$ appear as quartic curves $\{p^*\}$, and Ψ_2, $\Psi_{\bar{2}}$ as $\mathfrak{w}^{(13,1)}$, $\mathfrak{w}^{(1,13)}$, respectively, and joining corresponding points of Ψ_3, Ψ^* by lines. Ψ^* is the projective model of the system $|\Gamma+\Delta+4\Theta|$ on $W^*_{2,3}$; and by the usual device of comparing the sections of the locus so generated by primes through Ψ_3 and through Ψ^*, we find that

$$\Psi^* = -5(\Gamma+\Delta)+3\Theta+\Psi_3,$$

TABLE 10

	Γ section	Δ section	Θ section	Ψ_3 section	prime section	order
$W_{2,3}$	Γ	Δ	Θ	Ψ_3	$\Gamma+\Delta+4\Theta+\Psi_3$	97,546
Γ	$\mathfrak{c}$	$\mathfrak{c}+\mathfrak{d}$	$3(\mathfrak{c}+\mathfrak{d})+\mathfrak{s}_2$	$\mathfrak{s}_3$	$14\mathfrak{c}+13\mathfrak{d}+4\mathfrak{s}_2+\mathfrak{s}_3$	3,972
Δ	$\mathfrak{c}+\mathfrak{d}$	$\mathfrak{d}$	$3(\mathfrak{c}+\mathfrak{d})+\mathfrak{t}_{\bar{2}}$	$\mathfrak{t}_3$	$13\mathfrak{c}+14\mathfrak{d}+4\mathfrak{t}_{\bar{2}}+\mathfrak{t}_3$	3,972
Θ	$3(\mathfrak{c}+\mathfrak{d})+\mathfrak{s}_2$	$3(\mathfrak{c}+\mathfrak{d})+\mathfrak{t}_{\bar{2}}$	$9(\mathfrak{c}+\mathfrak{d})+3(\mathfrak{s}_2+\mathfrak{t}_{\bar{2}})$	$\mathfrak{u}$	$42(\mathfrak{c}+\mathfrak{d})+\mathfrak{u}+13(\mathfrak{s}_2+\mathfrak{t}_{\bar{2}})$	19,878
Ψ_3	$\mathfrak{s}_3$	$\mathfrak{t}_3$	$\mathfrak{u}$	$-3\mathfrak{u}+5(\mathfrak{s}_3+\mathfrak{t}_3)$	$\mathfrak{u}+6(\mathfrak{s}_3+\mathfrak{t}_3)$	9,990
$\mathfrak{c}$	0	W	$3W+S_2$	S_3	$13W+4S_2+S_3$	108
$\mathfrak{d}$	W	0	$3W+T_{\bar{2}}$	T_3	$13W+4T_{\bar{2}}+T_3$	108
$\mathfrak{u}$	$F_{2,3}+3(S_3+T_3)$	$G_{\bar{2},3}+3(S_3+T_3)$	$3(F_{2,3}+G_{\bar{2},3})+9(S_3+T_3)$	$-4(F_{2,3}+G_{\bar{2},3})+3(S_3+T_3)$	$9(F_{2,3}+G_{\bar{2},3})+45(S_3+T_3)$	3,942
$\mathfrak{s}_2$	S_2	$3W+S_2+T_{\bar{2}}$	$3S_2$	$F_{2,3}$	$F_{2,3}+3W+14S_2+T_{\bar{2}}$	264
$\mathfrak{t}_{\bar{2}}$	$3W+S_2+T_{\bar{2}}$	$T_{\bar{2}}$	$3T_{\bar{2}}$	$G_{\bar{2},3}$	$G_{\bar{2},3}+3W+S_2+14T_{\bar{2}}$	264
$\mathfrak{s}_3$	S_3	S_3+T_3	$F_{2,3}+3(S_3+T_3)$	$-3F_{2,3}+S_3-4T_3$	$F_{2,3}+15S_3+9T_3$	504
$\mathfrak{t}_3$	S_3+T_3	T_3	$G_{\bar{2},3}+3(S_3+T_3)$	$-3G_{\bar{2},3}-4S_3+T_3$	$G_{\bar{2},3}+9S_3+15T_3$	504
I	l_0	$l_0+l_1+3l_2+9l_3$	$3l_0$	0	$14l_0+l_1+3l_2+9l_3$	27
J	$l_0+l_1+3l_2+9l_3$	l_1	$3l_1$	0	$l_0+14l_1+3l_2+9l_3$	27
W	0	0	l_3	l_2	l_2+4l_3	5
S_2	0	l_3	0	l_0+5l_3	l_0+6l_3	7
$T_{\bar{2}}$	l_3	0	0	l_1+5l_3	l_1+6l_3	7
S_3	0	l_2	$l_0+3l_2+5l_3$	$-3l_0-3l_2-15l_3$	$l_0+9l_2+5l_3$	15
T_3	l_2	0	$l_1+3l_2+5l_3$	$-3l_1-3l_2-15l_3$	$l_1+9l_2+5l_3$	15
l_0	0	1	0	0	1	1
l_1	1	0	0	0	1	1
l_2	0	0	1	-3	1	1
l_3	0	0	0	1	1	1

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where of course Γ, Δ, Θ stand here for the images of these primals on $W^*_{2,2}$ in the generating line system. It is not immediately obvious that $W_{2,3}$ is not in fact this fivefold variety, the $\mathfrak{w}^{(13,1)}$, $\mathfrak{w}^{(1,13)}$ on Ψ^* being $\Psi_{(23)}$, $\Psi_{\overline{23}}$. In the paper on $W_{2,3}$ already referred to, however, I proved that there is on $W_{2,3}$ no surface unisecant to the lines $\{l_3\}$ and not meeting Φ_3; and it is clear that Ψ^*, if it existed on $W^*_{2,3}$, would trace on each threefold $\mathfrak{c}$, $\mathfrak{d}$ (images of $W_{2,3}$) precisely such a surface, namely S^*, T^*, respectively. The virtual linear system Ψ^* defined by (25·9) on $W^*_{2,3}$ has of course the properties of being unisecant to $\{l_3\}$ and having zero intersection with Ψ_3; but it is merely virtual, and contains no actual primals.

TABLE 11

image of $W^*_{2,3}$	in $\{l_4\}$ $W^*_{2,4}$	on Ψ_4 Ψ_4	image of	in $\{l_4\}$	on Ψ_4
Γ	Γ	$\mathfrak{X}_4$	S_i	$\mathfrak{x}_i$	$S_{i,4}$
Δ	Δ	$\mathfrak{Y}_4$	T_i	$\mathfrak{y}_i$	$T_{i,4}$
Θ	Θ	$\mathfrak{U}_4$	F_{ij}	$\mathfrak{s}_{ij}$	$F_{ij,4}$
Ψ_3	Ψ_3	$\Psi_{3,4}$	G_{ij}	$\mathfrak{t}_{ij}$	$G_{ij,4}$
			W	$W_{2,3}(\mathsf{P}_1)$	V_4
$\mathfrak{c}$	$\mathfrak{C}$	$\mathfrak{x}_4$	f_{ij}	S_{ij}	$f_{ij,4}$
$\mathfrak{d}$	$\mathfrak{D}$	$\mathfrak{y}_4$	g_{ij}	T_{ij}	$g_{ij,4}$
$\mathfrak{u}$	$\mathfrak{U}_3$	$\mathfrak{u}_{3,4}$	l_2	V_3	$p_{3,4}$
$\mathfrak{s}_i$	$\mathfrak{X}_i$	$\mathfrak{s}_{i,4}$	l_3	$W_{2,2}(\mathsf{P}_2)$	l_3
$\mathfrak{t}_i$	$\mathfrak{Y}_i$	$\mathfrak{t}_{i,4}$	point	l_4	point

$W^*_{2,4}$ need not be dealt with at length, but it can be pointed out briefly how the same methods can be applied to this case, and indeed to $W^*_{2,n+1}$ when $W^*_{2,n}$ has first been studied. $\mathfrak{C}$, $\mathfrak{D}$ are now fourfolds. We define on each of the Ψ primals (Ψ_2, $\Psi_{\overline{2}}$, Ψ_3, Ψ_4) the fourfolds $\mathfrak{X}$, $\mathfrak{Y}$ traced by Γ, Δ, the threefolds $\mathfrak{x}$, $\mathfrak{y}$ traced by $\mathfrak{C}$, $\mathfrak{D}$, and the surface V traced by the three-fold $W_{2,3}(\mathsf{P}_1)$; in each case distinguished by the suffix 2, $\overline{2}$, 3, 4 of the Ψ primal. Thus V_2, $V_{\overline{2}}$ are subsystems of the family $\{W_{2,2}(\mathsf{P}_2)\}$ of ruled quintics and V_3, V_4 are (on $\mathfrak{C}$ or $\mathfrak{D}$) the surfaces V, U of § 12. We denote further the traces of Θ on Ψ_3, Ψ_4, $\Psi_{(34)}$, $\Psi_{3,4}$ by $\mathfrak{U}_3$, $\mathfrak{U}_4$, $\mathfrak{u}_{(34)}$, $\mathfrak{u}_{3,4}$. Every subvariety of $W^*_{2,3}$ has an image in $\{l_4\}$, and one on Ψ_4, which is unisecant to $\{l_4\}$ and a birational image of $W^*_{2,3}$; these images are given in table 11, where i denotes any one of the condition symbols 2, $\overline{2}$, 3, and ij any one of the symbols (23), 2,3, $\overline{2}$,3, $2\overline{3}$; and $p_{3,4}$ is the sextic generating curve of $\Psi_{3,4}$ (the only one of the fourfold Ψ loci not generated by lines) which appears as the trace of $\Phi_{3,4}$ in figure 2, and (on either $\mathfrak{C}$ or $\mathfrak{D}$) is the curve on $W_{2,4}$ called k in § 12.

From the geometry on the various ruled surfaces we express all the curves f, g, and $p_{3,4}$ in terms of l_0, l_1. l_2, l_3, l_4; and from that on the fourfold Ψ loci we express all the surfaces F, G in terms of any one of each, say $I = F_{(234)}$, $J = G_{\overline{234}}$, which are those of lowest order, 81, and the surfaces S, T; and also all the threefold Ψ loci in terms of either $\mathfrak{u}_{(34)}$ or $\mathfrak{u}_{3,4}$ and the threefolds $\mathfrak{s}$, $\mathfrak{t}$. Further, since $\Psi_{(34)}$, $\Psi_{3,4}$ are birational models of $W^*_{2,2}$ on which the images of l_0, l_1, l_2 are of degrees 21, 21, 1, and 13, 13, 6, respectively, they are the projective models of the linear systems $|21(\Gamma+\Delta)+\Theta|$, $|13(\Gamma+\Delta)+6\Theta|$, respectively, from which the prime sections of any of their subvarieties can be obtained; and by comparing sections by primes through $\Psi_{(34)}$, $\Psi_{3,4}$, respectively, either of them, or any of its subvarieties, can be expressed in terms of the other and its subvarieties and the subvarieties of Ψ_3 ruled in $\{l_4\}$. These relations, together with those corresponding to (24·3, 4, 5, 6) in $\{l_4\}$ and on Ψ_4, enable us

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to express all the subvarieties of $W_{2,4}$ that have been defined in terms of a base consisting of five primals, eleven fourfolds, fourteen threefolds, eleven surfaces, and five curves, which can conveniently be taken to be

$$\Gamma, \quad \Delta, \quad \Theta, \quad \Psi_3, \quad \Psi_4;$$
$$\mathfrak{C}, \quad \mathfrak{D}, \quad \mathfrak{X}_2, \quad \mathfrak{Y}_{\bar{2}}, \quad \mathfrak{X}_3, \quad \mathfrak{Y}_3, \quad \mathfrak{X}_4, \quad \mathfrak{Y}_4, \quad \mathfrak{U}_3, \quad \mathfrak{U}_4, \quad \Psi_{(34)};$$
$$\mathfrak{x}_2, \quad \mathfrak{y}_{\bar{2}}, \quad \mathfrak{x}_3, \quad \mathfrak{y}_3, \quad \mathfrak{x}_4, \quad \mathfrak{y}_4, \quad \mathfrak{s}_{(23)}, \quad \mathfrak{t}_{\bar{2}\bar{3}}, \quad \mathfrak{s}_{2,4}, \quad \mathfrak{t}_{\bar{2},4}, \quad \mathfrak{s}_{(34)}, \quad \mathfrak{t}_{(34)}, \quad \mathfrak{u}_{(34)}, \quad W_{2,3}(\mathsf{P}_1).$$
$$S_{(23)}, \quad T_{\bar{2}\bar{3}}, \quad S_{2,4}, \quad T_{\bar{2},4}, \quad S_{(34)}, \quad T_{(34)}, \quad I, \quad J, \quad V_3, \quad V_4, \quad W_{2,2}(\mathsf{P}_2);$$
$$l_0, \quad l_1, \quad l_2, \quad l_3, \quad l_4.$$

The intersection table can be written down by the same methods as before; the prime sections of $W_{2,4}$ are easily seen to be the linear system $|\Gamma+\Delta+13\Theta+4\Psi_3+\Psi_4|$, and those of Ψ_4 are the linear system $|8(\mathfrak{X}_4+\mathfrak{Y}_4)+9\mathfrak{U}_4+\Psi_{3,4}|$; thus the virtual characteristic system of Ψ_4 is $|7(\mathfrak{X}_4+\mathfrak{Y}_4)-4\mathfrak{U}_4-3\Psi_{3,4}|$. This enables us to find the prime sections of any subvariety of Ψ_4, and its virtual intersection with Ψ_4, as its intersections with $|8(\Gamma+\Delta)+9\Theta+\Psi_3|$ and with $|7(\Gamma+\Delta)-4\Theta-3\Psi_3|$. This in turn enables us to find the order of each subvariety in turn, proceeding from lower dimensions to higher, and finally of $W^*_{2,4}$ itself, each as the sum of the orders of the different terms in its prime section. The calculations however are very laborious, though perfectly straightforward, and we shall not pursue them here.

It is obvious that with sufficient pertinacity we could carry on in this way as far as was worth while. All the equivalence and intersection problems on $W^*_{2,n}$ reduce to those on $W^*_{2,n'}$, for $n'<n$, applied either to a Ψ locus which is a birational transform of $W^*_{2,n'}$ or to a congruence of which $W^*_{2,n'}$ is a model. There are comparatively few general results that leap to the eye; the most important are that the prime sections of $W_{2,n}$ are the linear system

$$\left|\Gamma+\Delta+\tfrac{1}{2}(3^{n-1}-1)\,\Theta+\sum_{i=3}^{n}\tfrac{1}{2}(3^{n-i+1}-1)\,\Psi_i\right|$$

and that the intersection matrix of the base $\Gamma, \Delta, \Theta, \Psi_3, \ldots, \Psi_n$ for primals with the base $l_0, \ldots, l_n$ for curves is

$$\left|\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & -3 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & -3 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & -3 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix}\right|.$$

This makes it easy to write down the prime sections of any birational image of $W^*_{2,n}$ when we know the orders of the curve images of $l_0, \ldots, l_n$, and also of a variety generated by lines unisecant to such a birational image when we know the nature of the ruled surface images of $l_0, \ldots, l_n$, and of the unisecant variety.

26. Homogeneous t-invariants of a branch in S_3

We shall not carry the investigation of $W^*_{r,n}$ $(r \geqslant 3)$ very far. The mode of procedure is already clear, and we merely illustrate it by finding a few explicit formulae for $r=3$, as in more dimensions than this the notation becomes unmanageably complicated. We shall use a homogeneous co-ordinate system (u^w, u^x, u^y, u^z), and again specify the generic branch by the equations

$$u^\alpha = u_0^\alpha + u_1^\alpha t + u_2^\alpha t^2 + u_3^\alpha t^3 + \dots \tag{26·1}$$

which are exactly like (21·1) except that here of course $\alpha = w, x, y, z$. $(u_0^w, u_0^x, u_0^y, u_0^z)$, not all zero, are the co-ordinates of the origin P_0 of the branch; and we obtain an affine co-ordinate system with origin P_0 by writing

$$x = \frac{u^x}{u^w} - \frac{u_0^x}{u_0^w}, \quad y = \frac{u^y}{u^w} - \frac{u_0^y}{u_0^w}, \quad z = \frac{u^z}{u^w} - \frac{u_0^z}{u_0^w}. \tag{26·2}$$

We have to define the quadratic, cubic, and quartic determinants

$$q_{(ij)}^{(\alpha\beta)} = u_i^\alpha u_j^\beta - u_j^\alpha u_i^\beta, \quad q_{(\gamma\delta)(ij)} = \epsilon_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta$$

(so that $q_{(ij)}^{(wx)} = q_{(yz)(ij)}$, etc.);
$$D_{\delta(ijk)}^* = \epsilon_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta u_k^\gamma,$$
$$E_{(ijkl)}^* = \epsilon_{\alpha\beta\gamma\delta} u_i^\alpha u_j^\beta u_k^\gamma u_l^\delta,$$

between which we note the identities

$$\epsilon_{w\alpha\beta\gamma} q_{(ij)}^{(w\beta)} q_{(ik)}^{(w\gamma)} = u_i^w D_{\alpha(ijk)}^*, \tag{26·3}$$

$$\epsilon_{w\alpha\beta\gamma} q_{(ij)}^{(w\alpha)} q_{(ik)}^{(w\beta)} q_{(il)}^{(w\gamma)} = (u_i^w)^2 E_{(ijkl)}^*. \tag{26·4}$$

Now by substituting from (26·1) and ordinary division of the formal power series, we obtain an expansion of the form (13·1) with coefficients $a_i = p_i^x$, $b_i = p_i^y$, $c_i = p_i^z$ expressible as rational functions of the coefficients in (26·1); in fact, as in (21·3),

$$\left.\begin{aligned}
p_1^\alpha &= q_{(01)}^{(w\alpha)}/(u_0^w)^2,\\
p_2^\alpha &= (-u_1^w q_{(01)}^{(w\alpha)} + u_0^w q_{(02)}^{(w\alpha)})/(u_0^w)^3,\\
p_3^\alpha &= \{[(u_1^w)^2 - u_0^w u_2^w]\, q_{(01)}^{(w\alpha)} - u_0^w u_1^w q_{(02)}^{(w\alpha)} + (u_0^w)^2 q_{(03)}^{(w\alpha)}\}/(u_0^w)^4,\\
p_4^\alpha &= \{[-(u_0^w)^2 u_3^w + 2u_0^w u_1^w u_2^w - (u_1^w)^3]\, q_{(01)}^{(w\alpha)} + u_0^w[(u_1^w)^2 - u_0^w u_2^w]\, q_{(02)}^{(w\alpha)}\\
&\quad - (u_0^w)^2 u_1^w q_{(03)}^{(w\alpha)} + (u_0^w)^3 q_{(04)}^{(w\alpha)}\}/(u_0^w)^5,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{26·5}$$

where of course $\alpha = x, y, z$ only. Substituting these in the definitions of $D_{\alpha(ij)}$, $E_{(ijk)}$, and using (26·3) we have

$$\left.\begin{aligned}
D_{\alpha(12)} &= D_{\alpha(012)}^*/(u_0^w)^3,\\
D_{\alpha(13)} &= (-u_1^w D_{\alpha(012)}^* + u_0^w D_{\alpha(013)}^*)/(u_0^w)^4,\\
D_{\alpha(14)} &= \{[(u_1^w)^2 - u_0^w u_2^w]\, D_{\alpha(012)}^* - u_0^w u_1^w D_{\alpha(013)}^* + (u_0^w)^2 D_{\alpha(014)}^*\}/(u_0^w)^5,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{26·6}$$

and similarly using (26·4)

$$\left.\begin{aligned}
E_{(123)} &= E_{(0123)}^*/(u_0^w)^4,\\
E_{(124)} &= (u_1^w E_{(0123)}^* - u_0^w E_{(0124)}^*)/(u_0^w)^5,\\
E_{(134)} &= (u_2^w E_{(0123)}^* - u_1^w E_{(0124)}^* - u_0^w E_{(0134)}^*)/(u_0^w)^6,\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\} \tag{26·7}$$

It is at once clear that

$$\mathbf{u}^{\alpha} = u_0^{\alpha}, \quad \mathbf{q}^{(\alpha\beta)} = q_{(01)}^{(\alpha\beta)}, \quad \mathbf{D}_{\alpha}^{*} = D_{\alpha(012)}^{*}, \quad \mathbf{E}^{*} = E_{(0123)}^{*}$$

are t-invariant tensors of ranks $0, 1, 2, 3$, weights $0, 1, 3, 6$, and degrees $1, 2, 3, 4$, respectively; the first three of these are the homogeneous co-ordinates of the origin P_0, the tangent line $\mathsf{P}_0\mathsf{P}_1$, and the osculating plane $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2$ of the branch; while $\mathbf{E}^{*} = 0$ is the condition for the points $\mathsf{P}_0\mathsf{P}_1\mathsf{P}_2\mathsf{P}_3$ to be coplanar, i.e. for the origin to be a point of stationary osculation. But substituting the values (26·5, 6, 7) in the expression for any t-invariant of (13·1) as a (p, D, E) form, we obtain a fraction whose denominator is a power of $\mathbf{u}^{w} = u_0^{w}$, and whose numerator is a form in the u_i^{w}'s, $q_{(ij)}^{(w\alpha)}$'s, $D_{\alpha(ijk)}^{*}$'s, and $E_{(ijkl)}^{*}$'s ($\alpha = x, y, z$ only); and the numerators so arising are certain components of certain t-invariant homogeneous tensors; those components namely in which every separate contravariant index, one out of every antisymmetric pair of contravariant indices, and no covariant index, is equal to w. Thus the remaining principal t-invariant $\mathbf{G}_{\beta}^{\alpha}$ of rank 3, and the five principal t-invariant tensors of rank 4 found in § 13, give us

$$\mathbf{G}_{\beta}^{\alpha} = \frac{\mathbf{G}_{\beta}^{*w(w\alpha)}}{(\mathbf{u}^{w})^{6}}, \quad \mathbf{I}_{\gamma}^{\alpha\beta} = \frac{\mathbf{I}_{\gamma}^{*ww(w\alpha)(w\beta)}}{(\mathbf{u}^{w})^{9}}, \quad \mathbf{J}_{\gamma,\delta}^{\alpha\beta} = \frac{\mathbf{J}_{\gamma,\delta}^{*ww(w\alpha)(w\beta)}}{(\mathbf{u}^{w})^{12}},$$

$$\mathbf{S}^{\alpha} = \frac{\mathbf{S}^{*w(w\alpha)}}{(\mathbf{u}^{w})^{7}}, \quad \mathbf{T}_{\beta}^{\alpha} = \frac{\mathbf{T}_{\beta}^{*(w\alpha)}}{(\mathbf{u}^{w})^{9}}, \quad \mathbf{U}^{\alpha\beta} = \frac{\mathbf{U}^{*ww(w\alpha)(w\beta)}}{(\mathbf{u}^{w})^{10}},$$

where

$$\mathbf{G}_{\delta}^{*\alpha(\beta\gamma)} = (u_1^{\alpha} q_{(01)}^{(\beta\gamma)} - 2u_0^{\alpha} q_{(02)}^{(\beta\gamma)})\, D_{\delta(012)}^{*} + u_0^{\alpha} q_{(01)}^{(\beta\gamma)} D_{\delta(013)}^{*},$$

$$\begin{aligned}\mathbf{I}_{\kappa}^{*\alpha\beta(\gamma\delta)(\eta\theta)} = {} & \{\tfrac{1}{2}(u_0^{\alpha} u_2^{\beta} + 2u_1^{\alpha} u_1^{\beta} + u_2^{\alpha} u_0^{\beta})\, q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} - \tfrac{5}{4}(u_0^{\alpha} u_1^{\beta} + u_1^{\alpha} u_0^{\beta})\, (q_{(01)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)}) \\ & - u_0^{\alpha} u_0^{\beta} (q_{(01)}^{(\gamma\delta)} q_{(03)}^{(\eta\theta)} - 5q_{(02)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(03)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)})\}\, D_{\kappa(012)}^{*} \\ & + \{(u_0^{\alpha} u_1^{\beta} + u_1^{\alpha} u_0^{\beta})\, q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} - \tfrac{3}{2} u_0^{\alpha} u_0^{\beta} (q_{(01)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)})\}\, D_{\kappa(013)}^{*} \\ & + u_0^{\alpha} u_0^{\beta} q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} D_{\kappa(014)}^{*},\end{aligned}$$

$$\begin{aligned}\mathbf{J}_{\kappa,\lambda}^{*\alpha\beta(\gamma\delta)(\eta\theta)} = {} & \{\tfrac{1}{2}(u_0^{\alpha} u_2^{\beta} - 2u_1^{\alpha} u_1^{\beta} + u_2^{\alpha} u_0^{\beta})\, q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} \\ & + \tfrac{3}{4}(u_0^{\alpha} u_1^{\beta} + u_1^{\alpha} u_0^{\beta})\, (q_{(01)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)}) - u_0^{\alpha} u_0^{\beta} (q_{(01)}^{(\gamma\delta)} q_{(03)}^{(\eta\theta)} + 3q_{(02)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} \\ & + q_{(03)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)})\}\, D_{\kappa(012)}^{*} D_{\lambda(012)}^{*} - (u_0^{\alpha} u_1^{\beta} + u_1^{\alpha} u_0^{\beta})\, q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\xi\theta)} D_{\kappa(012)}^{*} D_{\lambda(013)}^{*} \\ & + \tfrac{1}{2} u_0^{\alpha} u_0^{\beta} (q_{(01)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)})\, (4D_{\kappa(012)}^{*} D_{\lambda(013)}^{*} + D_{\kappa(013)}^{*} D_{\lambda(012)}^{*}) \\ & + u_0^{\alpha} u_0^{\beta} q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} (D_{\kappa(014)}^{*} D_{\lambda(012)}^{*} - 2D_{\kappa(013)}^{*} D_{\lambda(013)}^{*}).\end{aligned}$$

(satisfying

$$\begin{aligned}\mathbf{J}_{\kappa,\lambda}^{*\alpha\beta(\gamma\delta)(\eta\theta)} = \mathbf{D}_{\lambda}^{*}\, \mathbf{I}_{\kappa}^{*\alpha\beta(\gamma\delta)(\eta\theta)} - \tfrac{1}{2}(& \mathbf{G}_{\kappa}^{*\alpha(\gamma\delta)} \mathbf{G}_{\lambda}^{*\beta(\eta\theta)} \\ & + \mathbf{G}_{\kappa}^{*\alpha(\eta\theta)} \mathbf{G}_{\lambda}^{*\beta(\gamma\delta)} + \mathbf{G}_{\kappa}^{*\beta(\gamma\delta)} \mathbf{G}_{\lambda}^{*\alpha(\eta\theta)} + \mathbf{G}_{\kappa}^{*\beta(\eta\theta)} \mathbf{G}_{\lambda}^{*\alpha(\gamma\delta)})\end{aligned}$$

which seems to be the most extended version of (5·4) that we reach in the course of this work).

$$\mathbf{S}^{*\alpha(\beta\gamma)} = (2u_1^{\alpha} q_{(01)}^{(\beta\gamma)} - 3u_0^{\alpha} q_{(02)}^{(\beta\gamma)})\, E_{(0123)}^{*} + u_0^{\alpha} q_{(01)}^{(\beta\gamma)} E_{(0124)}^{*},$$

$$\mathbf{T}_{\gamma}^{*(\alpha\beta)} = (q_{(02)}^{(\alpha\beta)} D_{\gamma(012)}^{*} - 2q_{(01)}^{(\alpha\beta)} D_{\gamma(013)}^{*})\, E_{(0123)}^{*} - q_{(01)}^{(\alpha\beta)} D_{\gamma(012)}^{*} E_{(0124)}^{*},$$

$$\begin{aligned}\mathbf{U}^{*\alpha\beta(\gamma\delta)(\eta\theta)} = {} & \{\tfrac{1}{2}(-u_0^{\alpha} u_2^{\beta} + 2u_1^{\alpha} u_1^{\beta} - u_2^{\alpha} u_0^{\beta})\, q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} - \tfrac{1}{2}(u_0^{\alpha} u_1^{\beta} + u_1^{\alpha} u_0^{\beta})\, (q_{(01)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)}) \\ & - u_0^{\alpha} u_0^{\beta} (q_{(01)}^{(\gamma\delta)} q_{(03)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(03)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)})\}\, E_{(0123)}^{*} \\ & + \{\tfrac{1}{2}(u_0^{\alpha} u_1^{\beta} + u_1^{\alpha} u_0^{\beta})\, q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} - u_0^{\alpha} u_0^{\beta} (q_{(01)}^{(\gamma\delta)} q_{(02)}^{(\eta\theta)} + q_{(02)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)})\}\, E_{(0124)}^{*} \\ & + u_0^{\alpha} u_0^{\beta} q_{(01)}^{(\gamma\delta)} q_{(01)}^{(\eta\theta)} E_{(0134)}^{*}.\end{aligned}$$

It is seen that $\mathbf{G}^{*\alpha(\beta\gamma)}_{\delta}$ is a t-invariant tensor of rank 3, weight 5, homogeneous degree 6, and having 96 components, as each of α, δ has the four values w, x, y, z, and the antisymmetric pair $(\beta\gamma)$ has six values. Similarly the five t-invariant tensors of rank 4 have respective weights 7, 10, 8, 11, 10, and degrees 9, 12, 7, 9, 10, and have respectively 840, 3360, 24, 24, and 210 components. (The pairs of indices enclosed in brackets are antisymmetric, but $\mathbf{I}^{*\alpha\beta(\gamma\delta)(\eta\theta)}_{\kappa}$, $\mathbf{J}^{*\alpha\beta(\gamma\delta)(\eta\theta)}_{\kappa,\lambda}$, $\mathbf{U}^{*\alpha\beta(\gamma\delta)(\eta\theta)}$ are symmetric with respect to α, β, and also with respect to the pairs $(\gamma\delta)$, $(\eta\theta)$.)

In four dimensions we shall get obviously a very similar set of t-invariant tensors in which, however, the cubic determinants D^* will have two covariant (or of course three contravariant) indices, and the quartic determinants E^* will have one covariant index; also, we shall have further t-invariants, all vanishing when the branch lies in S_3, containing the quintic determinants

$$F^*_{(ijklm)} = \epsilon_{\alpha\beta\gamma\delta\xi}\, u^\alpha_i u^\beta_j u^\gamma_k u^\delta_l u^\xi_m,$$

starting with $\mathbf{F}^* = F^*_{(01234)}$ of rank 4.

27. Parametrization of $W^*_{3,n}$

We can now of course substitute these values of the affine t-invariants in terms of the homogeneous in the parametrization of $W_{3,n}$. We obtain in the first instance fractions whose denominators are powers of $\mathbf{u}^w$; and multiplying throughout by $(\mathbf{u}^w)^{3n}$ we obtain a set of monomials, isobaric and homogeneous, which are certain components of a set of t-invariant tensors—those components namely in which each separate contravariant index, one of each antisymmetric pair of contravariant indices, and no covariant index, is w. Adjoining all the remaining components of these tensors, and taking the whole set of t-invariants so obtained as homogeneous co-ordinates in projective space of suitably high dimensions, we obtain the generic point of $W^*_{3,n}$.

Thus for $W^*_{3,1}$ we have clearly

$$X^{\alpha(\beta\gamma)} = \mathbf{u}^\alpha \mathbf{q}^{(\beta\gamma)}, \tag{27·1}$$

satisfying the linear relation

$$X^{\alpha(\beta\gamma)} + X^{\beta(\gamma\alpha)} + X^{\gamma(\alpha\beta)} = 0.$$

Similarly from (14·1) we obtain in the first instance

$$X^\alpha_\beta = \mathbf{q}^{(w\alpha)}\mathbf{D}^*_\beta/(\mathbf{u}^w)^5, \quad Y^{\alpha\beta\gamma\delta} = \mathbf{q}^{(w\alpha)}\mathbf{q}^{(w\beta)}\mathbf{q}^{(w\gamma)}\mathbf{q}^{(w\delta)}/(\mathbf{u}^w)^8;$$

whence multiplying by $(\mathbf{u}^w)^9$ and adjoining all the remaining components of these two tensors we obtain

$$\left.\begin{aligned} X^{\alpha_1\alpha_2\alpha_3\alpha_4(\beta\gamma)}_\delta &= \mathbf{u}^{\alpha_1}\mathbf{u}^{\alpha_2}\mathbf{u}^{\alpha_3}\mathbf{u}^{\alpha_4}\mathbf{q}^{(\beta\gamma)}\mathbf{D}^*_\delta, \\ Y^{\alpha(\beta_1\gamma_1)(\beta_2\gamma_2)(\beta_3\gamma_3)(\beta_4\gamma_4)} &= \mathbf{u}^\alpha\,\mathbf{q}^{(\beta_1\gamma_1)}\mathbf{q}^{(\beta_2\gamma_2)}\mathbf{q}^{(\beta_3\gamma_3)}\mathbf{q}^{(\beta_4\gamma_4)} \end{aligned}\right\} \tag{27·2}$$

as the parametrization of $W^*_{3,2}$. The vanishing of all co-ordinates X gives the locus $\Psi_{\bar{2}}$, which is a birational model of $W^*_{3,1}$; while that of all co-ordinates Y gives the locus Ψ_2, which is a birational model of the so-called flag manifold, model of the aggregate of figures in S_3 consisting of a point, a line through it, and a plane through the line. Given values of $\mathbf{u}^\alpha$, $\mathbf{q}^{(\beta\gamma)}$ determine a line on Ψ_2 and a point on $\Psi_{\bar{2}}$, and $W_{3,2}$ is generated by the ∞^5 planes $W_{3,1}(\mathsf{P}_0\mathsf{P}_1)$ joining a point and line so corresponding. This is in fact easily seen to be precisely Longo's model of $W^*_{3,2}$.

Similarly, on substituting in (14·6) the values of the various tensors found in the last section, we find that the nine monomial tensors in (14·6) become fractions with denominators $(\mathbf{u}^w)^{19}$, $(\mathbf{u}^w)^{22}$, $(\mathbf{u}^w)^{15}$, $(\mathbf{u}^w)^{18}$, $(\mathbf{u}^w)^{14}$, $(\mathbf{u}^w)^{17}$, $(\mathbf{u}^w)^{20}$, $(\mathbf{u}^w)^{23}$, $(\mathbf{u}^w)^{26}$; thus multiplying throughout by $(\mathbf{u}^w)^{27}$ and adjoining all the remaining components of the tensors as before we obtain

$$
\begin{aligned}
X^{\alpha_1\dots\alpha_8(\beta_1\gamma_1)\dots(\beta_5\gamma_5),\,\lambda(\mu\nu)}_{\delta,\,\kappa} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_8}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_5\gamma_5)}\mathbf{D}^*_\delta\,\mathbf{G}^{*\lambda(\mu\nu)}_\kappa,\\
X^{\alpha_1\dots\alpha_5(\beta_1\gamma_1)\dots(\beta_8\gamma_8),\,\lambda(\mu\nu)}_{\kappa} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_5}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_8\gamma_8)}\mathbf{G}^{*\lambda(\mu\nu)}_\kappa,\\
Y^{\alpha_1\dots\alpha_{12}(\beta_1\gamma_1)\dots(\beta_4\gamma_4)}_{\delta} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_{12}}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_4\gamma_4)}\mathbf{D}^*_\delta\,\mathbf{E}^*,\\
Y^{\alpha_1\dots\alpha_9(\beta_1\gamma_1)\dots(\beta_7\gamma_7)} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_9}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_7\gamma_7)}\mathbf{E}^*,\\
Z^{\alpha_1\dots\alpha_{13}(\beta\gamma)}_{\delta_1\dots\delta_4} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_{13}}\mathbf{q}^{(\beta\gamma)}\mathbf{D}^*_{\delta_1}\dots\mathbf{D}^*_{\delta_4},\\
Z^{\alpha_1\dots\alpha_{10}(\beta_1\gamma_1)\dots(\beta_4\gamma_4)}_{\delta_1\delta_2\delta_3} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_{10}}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_4\beta_4)}\mathbf{D}^*_{\delta_1}\mathbf{D}^*_{\delta_2}\mathbf{D}^*_{\delta_3},\\
Z^{\alpha_1\dots\alpha_7(\beta_1\gamma_1)\dots(\beta_7\gamma_7)}_{\delta_1\delta_2} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_7}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_7\gamma_7)}\mathbf{D}^*_{\delta_1}\mathbf{D}^*_{\delta_2},\\
Z^{\alpha_1\,\alpha_4(\beta_1\gamma_1)\dots(\beta_{10}\gamma_{10})}_{\delta} &= \mathbf{u}^{\alpha_1}\dots\mathbf{u}^{\alpha_4}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_{10}\gamma_{10})}\mathbf{D}^*_\delta,\\
Z^{\alpha(\beta_1\gamma_1)\dots(\beta_{13}\gamma_{13})} &= \mathbf{u}^{\alpha}\mathbf{q}^{(\beta_1\gamma_1)}\dots\mathbf{q}^{(\beta_{13}\gamma_{13})}
\end{aligned}
$$

as the parametrization of $W^*_{3,3}$. The various Ψ loci on this can of course be parametrized by applying a similar process to the parametrization of the corresponding Φ loci on $W_{3,3}$; they can, however, be identified without this, the co-ordinate tensors which vanish on each Ψ locus being those which arise by the substitution just described from those which vanish on the corresponding Φ locus.

In particular there are on $W^*_{3,3}$ four birational images of the flag manifold of S_3. The minimum model of this, parametrized by the single co-ordinate tensor $\mathbf{u}^\alpha\mathbf{q}^{(\beta\gamma)}\mathbf{D}^*_\delta$, has on it three congruences of ∞^5 lines, say $\{l\}$, $\{m\}$, $\{n\}$, loci of images of 'flags' in which the line and plane, the point and plane, and the point and line respectively are fixed. On the locus Ψ_2 on $W^*_{3,2}$ the images of the lines $\{l\}$ are quartic curves, those of the other two systems are lines, and it is the lines $\{n\}$ that are joined by planes to the points of $\Psi_{\bar{2}}$.

On $W^*_{3,3}$ the loci $\Psi_{2,3,\bar{3}}$, $\Psi_{(23)3}$, $\Psi_{(2\bar{3})}$, and $\Psi_{\bar{2},3}$ are given by the vanishing of all the co-ordinates except

$$
X^{\alpha_1\dots\alpha_8(\beta_1\gamma_1)\dots(\beta_5\gamma_5),\,\lambda(\mu\nu)}_{\delta,\,\kappa},\quad Y^{\alpha_1\dots\alpha_{12}(\beta_1\gamma_1)\dots(\beta_4\gamma_4)}_{\delta},
$$
$$
Z^{\alpha_1\dots\alpha_{13}(\beta\gamma)}_{\delta_1\dots\delta_4},\quad X^{\alpha_1\dots\alpha_5(\beta_1\gamma_1)\dots(\beta_8\gamma_8),\,\lambda(\mu\nu)}_{\kappa},
$$

respectively, and are thus birational models of the flag manifold, on which the images of the lines $\{l\}$, $\{m\}$, $\{n\}$ are of orders 9, 6, 2; 12, 4, 1; 13, 1, 4; 6, 9, 1. Exactly as on $W_{3,3}$ the loci $\Psi_{2,3}$, $\Psi_{(23)}$, $\Psi_{2,\bar{3}}$ and $\Psi_{3,\bar{3}}$ are generated each by ∞^6 lines joining corresponding points of two of these four birational images of the same variety, and Ψ_2 is generated by ∞^6 planes joining corresponding points of the first three. $\Psi_{\bar{2}}$ on the other hand is generated by ∞^5 planes, each joining a line n of $\Psi_{\bar{2},3}$ to the corresponding point of $\Psi_{\bar{2}\bar{3}}$, which is itself a birational image of $W^*_{3,1}$, being given by the vanishing of all the co-ordinates except $Z^{\alpha(\beta_1\gamma_1)\dots(\beta_{13}\gamma_{13})}$. Ψ_3, $\Psi_{\bar{3}}$ do not lend themselves to equally simple description; their equations are the vanishing of all co-ordinates $Z^{\dots}_{\dots}$ and that of all co-ordinates $Y^{\dots}_{\dots}$, respectively.

The most immediately striking feature of these loci $W^*_{3,n}$ is of course the absence of the symmetry between the conditions 2, $\bar{2}$, which was so conspicuous a feature of $W^*_{2,n}$ corresponding there to the duality transformation. This lack of symmetry in $W^*_{3,n}$ (and *a fortiori* in all $W^*_{r,n}$, $r \geqslant 3$) reflects the fact that for $r \geqslant 3$ there is no duality transformation on

$W_{r,n}$. Every branch in S_r has of course a well-defined dual branch, but the sequence $\mathsf{P}_0 \ldots \mathsf{P}_n$ on the one does not determine the sequence $\mathsf{P}'_0 \ldots \mathsf{P}'_n$ on the other, but only $\mathsf{P}'_0 \ldots \mathsf{P}'_{n-r+2}$, i.e. the duals of all branches through $\mathsf{P}_0 \ldots \mathsf{P}_n$ have no point common to all of them beyond P'_{n-r+2}, as is easily seen by arguments analogous to those at the beginning of § 23. This is further underlined by the fact that conditions dual to each other in S_3 familiarly do not generally involve the same number of points, e.g. not 2 and $\bar{2}$, but 2 and $\bar{3}$ are dual, i.e. the dual of a cuspidal branch is one on which the first four points are coplanar.

It is by now obvious that the above methods suffice for the parametrization of $W^*_{r,n}$, for all values of r and n.

Note 1. The number of elements in a base

If $b_{h,r,n}$ is the number of elements in a base for varieties of h dimensions on $W_{r,n}$ (or $W^*_{r,n}$), since $W_{r,n}$ is a fibre space of S_{r-1}'s over $W_{r,n-1}$ (and $W^*_{r,n}$ similarly over $W^*_{r,n-1}$), every such h-dimensional variety either meets ∞^h of these S_{r-1}'s in points, ∞^{h-1} of them in lines, ..., or is generated by ∞^{h-r-1} of them; so that

$$b_{h,r,n} = \sum_{i=0}^{r-1} b_{h-i,r,n-1},$$

where of course we put $b_{h,r,n} = 0$ if h is negative or greater than the dimension N of $W_{r,n}$ (or $W^*_{r,n}$). Hence defining

$$\phi_{r,n}(t) = \sum_{i=0}^{N} b_{h,r,n} t^i$$

we see that

$$\phi_{r,n}(t) = \left(\sum_{i=0}^{r-1} t^i\right) \phi_{r,n-1}(t).$$

Since moreover $W_{r,0}$ is a single point, and $W^*_{r,0}$ is S_r, we have for $W_{r,n}$

$$\phi_{r,n}(t) = \left(\sum_{i=0}^{r-1} t^i\right)^n$$

and for $W^*_{r,n}$

$$\phi_{r,n}(t) = \left(\sum_{i=0}^{r-1} t^i\right)^n \sum_{i=0}^{r} t^i.$$

These formulae give all the base numbers of $W_{r,n}$ and $W^*_{r,n}$.

Note 2. A simplified parametrization of $W_{2,n}$

In the note (Du Val 1961) already several times referred to, $W_{2,3}$ was parametrized rationally in terms of three independent parameters. It is clear that this can be done for any $W_{2,n}$ in terms of the parameters $\mu_1, \ldots, \mu_n$ defined in § 4; but simpler expressions, at least for the first few values of n, are obtained in terms of the parameters $\lambda_1, \ldots, \lambda_n$ which we now define recursively: $\lambda_1 = \mu_1 = \mathbf{b}/\mathbf{a}$; and λ_n is obtained from λ_{n-1} by the dilating substitution, preceded by the linear transformation $(x, y) \to (y, -x)$; i.e. by replacing, in each t-invariant (a, D) form, a_i by d_i, and as before D_{ij} by Δ_{ij}.

It is easily verified that

$$\lambda_2 = \frac{\mathbf{a}^3}{\mathbf{D}}, \quad \lambda_3 = \frac{\mathbf{D}^3}{\mathbf{a}^4 \mathbf{G}}, \quad \lambda_4 = \frac{\mathbf{a}^7 \mathbf{G}^3}{\mathbf{D}^4 \mathbf{J}}, \quad \lambda_5 = \frac{\mathbf{D}^7 \mathbf{J}^3}{\mathbf{a}^{11} \mathbf{G}^4 \mathbf{R}}, \quad \ldots,$$

and generally

$$\lambda_n = \prod_{i=1}^{n} \mathbf{E}_{(i)}^{\rho_{n-i}},$$

where $\mathbf{E}_{(1)} = \mathbf{a}$, $\mathbf{E}_{(2)} = \mathbf{D}$, and generally $\mathbf{E}_{(n)}$ is the highest in weight of the principal t-invariant (a, D) forms of rank n, denoted in § 5 by $(1, 2, \ldots, 2^{n-4}, 0)$; and the exponents $\rho_0, \rho_1, \ldots$ (alternately negative and positive) are defined by $\rho_0 = -1$, $\rho_1 = 3$, and for $n \geqslant 2$

$$\rho_n + \rho_{n-1} - \rho_{n-2} = 0,$$

so that

$$\sum_{i=0}^{\infty} \rho_i t^i = \frac{2t-1}{1+t-t^2}.$$

No demonstrative use has been made of these parameters in this paper, partly because a similarly simple set of algebraically independent parameters for $W_{r,n}$ $(r \geqslant 3)$ or for $W^*_{r,n}$ does not seem to be obviously available, but even more because their usefulness in simplifying the geometry of $W_{2,n}$ depends largely on the fact that:

If $\mathbf{F}$ is any principal t-invariant of rank n and of weight s, and $s' = 5 \times 2^{n-2}$ is the weight of $\mathbf{E}_{(n)}$, then $\mathbf{a}^{s'-s}\mathbf{F}/\mathbf{E}_{(n)}$, which is obviously a rational function of $(\lambda_1, \ldots, \lambda_n)$, is in fact a polynomial in $(\lambda_1, \ldots, \lambda_n)$.

Of this theorem I have been unable to construct a proof, though it has been verified for $n \leqslant 5$, and the outline of an inductive proof can be dimly discerned. Without it there seemed to be little point in introducing these parameters.

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